ON THE GENERALIZED OF HARMONIC AND BI-HARMONIC MAPS

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Abstract. In this note, we extend the definition of harmonic and biharmonic maps between two Riemannian manifolds, and we present some properties for \( f \)-harmonic maps and \( f \)-biharmonic maps.

1. \( f \)-HARMONIC MAPS

Definition 1.1. Consider a smooth map \( \varphi : (M^m, g) \longrightarrow (N^n, h) \) between Riemannian manifolds and \( f : (x, y) \in M \times N \longrightarrow f(x, y) \in (0, +\infty) \) be a smooth positive function. The \( f \)-energy functional of \( \varphi \) is defined by

\[
E_f(\varphi) = \frac{1}{2} \int_M f(x, \varphi(x)) |d\varphi|^2 v_g
\]

(or over any compact subset \( K \subset M \)).

A map is called \( f \)-harmonic if it is a critical point of the \( f \)-energy functional over any compact subset of \( M \).

Remark 1.1. :
- Definition 1.1, is a natural generalization of harmonic map ([2], [6], [7]) and \( f \)-harmonic map ([5], [10]).
- Definition 1.1, is also a generalization of \( p \)-harmonic map ([3]) and \( F \)-harmonic map [1]), when \( \varphi \) has no critical points.

1.1. The first variation of the \( f \)-energy. Let \( \varphi : (M^m, g) \longrightarrow (N^n, h) \) be a smooth map, we denote by :

\[
\tau(\varphi) = \text{trace}_g \nabla d\varphi
\]

the tension field of \( \varphi \), and \( e(\varphi) = \frac{1}{2} |d\varphi|^2 \) the energy density of \( \varphi \) (for more detail, see [2], [4], [6] and [7]).
Theorem 1.1. Let \( I = (-\epsilon, \epsilon) \subset \mathbb{R} \) and \( \{ \varphi_t \}_{t \in I} \) be a smooth variation of \( \varphi \). Then

\[
\frac{d}{dt} E_f(\varphi_t) \bigg|_{t=0} = -\int_M h(\tau_f(\varphi), v) g, \tag{1.2}
\]

where

\[
\tau_f(\varphi) = f \varphi \tau(\varphi) + d\varphi(\text{grad}^M f \varphi) - e(\varphi)(\text{grad}^N f) \circ \varphi, \tag{1.3}
\]

\( f \varphi \) is the smooth function \( x \in M^m \to f(\varphi(x)) \in (0, +\infty) \), and \( v = \frac{\partial \varphi}{\partial t} \bigg|_{t=0} \) denotes the variation vector field of \( \{ \varphi_t \}_{t \in I} \).

Proof:

Let \( \phi : I \times M \to N \) be a smooth map satisfying for all \( t \in I \) and all \( x \in M \)

\[
\phi(t, x) = \varphi_t(x),
\]

and

\[
\phi(0, x) = \varphi(x).
\]

The variation vector field \( v \in \Gamma(\varphi^{-1}TN) \) associated to the variation \( \{ \varphi_t \}_{t \in I} \) is given for all \( x \in M \), by

\[
v(x) = d_{(0, x)} \phi(\frac{\partial}{\partial t}),
\]

We have

\[
\frac{d}{dt} E_f(\varphi_t) \bigg|_{t=0} = \frac{1}{2} \int_M \left\{ \frac{\partial}{\partial t} f(x, \varphi_t(x)) \right\} |d_x \varphi_t|^2 \bigg|_{(0, x)} \nu_g
\]

\[
= \frac{1}{2} \int_M \left\{ \frac{\partial}{\partial t} f(x, \varphi_t(x)) \right\} |d_x \varphi|^2
\]

\[
+ f(x, \varphi_t(x)) \frac{\partial}{\partial t} |d_x \varphi_t|^2 \bigg|_{(0, x)} \nu_g. \tag{1.4}
\]

First, note that:

\[
\frac{\partial}{\partial t} f(x, \varphi_t(x)) \bigg|_{(0, x)} = \frac{\partial}{\partial t} f(x, \phi(t, x)) \bigg|_{(0, x)} = d_{x, \phi(t, x)} f(0, v(x)) = d_{\varphi(x)} f_x(v(x)),
\]

where \( f_x \) is the smooth function \( y \in N \to f_x(y) = f(x, y) \in (0, +\infty) \). Hence

\[
\frac{1}{2} \frac{\partial}{\partial t} f(x, \varphi_t(x)) \bigg|_{(0, x)} |d_x \varphi|^2 = d_{\varphi(x)} f_x(v(x)) e(\varphi)_x
\]

\[
= h((\text{grad}^N f_x)_\varphi(x), v(x)) e(\varphi)_x
\]

\[
= h(e(\varphi)_x (\text{grad}^N f_x)_\varphi(x), v(x)). \tag{1.5}
\]

Other hand:

Let \( \{ e_i \}_{i=1}^m \) be an orthonormal frame with respect to \( g \) on \( M \), such that \( \nabla_{e_i} e_j = 0 \), at \( x \in M \) for all \( i, j = 1, \ldots, m \). From equality

\[
d_x \varphi_t(e_i) = d_{(t, x)} \phi(0, e_i),
\]
Summing over the index $i$, we obtain
\[
\frac{1}{2} \frac{\partial}{\partial t} |d_x \varphi_t|^2 \bigg|_{(0,x)} = \frac{1}{2} \frac{\partial}{\partial t} h(d_x \varphi_t(e_i), d_x \varphi_t(e_i)) \bigg|_{(0,x)} \\
= \frac{1}{2} \frac{\partial}{\partial t} h(\phi(0, e_i), d\phi(0, e_i)) \bigg|_{(0,x)} \\
= h(\nabla^\phi \frac{\partial}{\partial t}, d\phi(0, e_i)) \bigg|_{(0,x)} \\
= h(\nabla^\phi_{(0,e_i)} \frac{\partial}{\partial t}, d\phi(0, e_i)) \bigg|_{(0,x)} \\
= (0, e_i)(h(\phi(0, e_i), d\phi(0, e_i))) \bigg|_{(0,x)} \\
= -h(\phi(0, e_i), \nabla^\phi_{(0,e_i)} d\phi(0, e_i)) \bigg|_{(0,x)} \\
= e_i(h(v, d\varphi(0, e_i)))_x - h(v, \tau(\varphi))_x.
\]

Let $X$ be a vector field with compact support on $M$, such that for any vector field $Y$ on $M$, we have
\[
g(X, Y) = h(v, d\varphi(Y)),
\]
then
\[
\frac{1}{2} \frac{\partial}{\partial t} |d_x \varphi_t|^2 \bigg|_{(0,x)} = (\text{div} X)_x - h(v, \tau(\varphi))_x.
\]
So :
\[
\frac{1}{2} f(x, \varphi(x)) \frac{\partial}{\partial t} |d_x \varphi_t|^2 \bigg|_{(0,x)} = \left[ f_\varphi \text{div} X - h(v, f_\varphi \tau(\varphi)) \right]_x \\
(1.6) = \left[ \text{div} (f_\varphi X) - h(v, d\varphi(\text{grad}^M f_\varphi)) - h(v, f_\varphi \tau(\varphi)) \right]_x.
\]
Substituting (1.5) and (1.6) in (1.4), and consider the divergence theorem (see [2]), we obtain
\[
\frac{d}{dt} E_f(\varphi_t) \bigg|_{t=0} = \int_M h(e(\varphi)_x (\text{grad}^N f_\varphi)_x - d\varphi(\text{grad}^M f_\varphi)_x \\
- f(x, \varphi(x)) \tau(\varphi)_x, v(x)) v_y.
\]

**Definition 1.2.** The field $\tau_f(\varphi)$ defined by :
\[
(1.7) \quad \tau_f(\varphi) = f_\varphi \tau(\varphi) + d\varphi(\text{grad}^M f_\varphi) - e(\varphi)(\text{grad}^N f) \circ \varphi \\
= \text{trace}_g \nabla f_\varphi d\varphi - e(\varphi)(\text{grad}^N f) \circ \varphi.
\]
is called the $f$-tension field of $\varphi$.

From Theorem 1.1, we deduce

**Theorem 1.2.** Let $\varphi : (M^m, g) \rightarrow (N^n, h)$ be a smooth map. Then $\varphi$ is $f$-harmonic, if and only if
\[
\tau_f(\varphi) = f_\varphi \tau(\varphi) + d\varphi(\text{grad}^M f_\varphi) - e(\varphi)(\text{grad}^N f) \circ \varphi = 0.
\]

**Particular Cases**
(1) If \( f = 1 \), then \( \tau_f(\varphi) = \tau(\varphi) \), is the natural tension field of \( \varphi \) (see [2], [6], [7]).

(2) Let \( f_1 : M \to (0, +\infty) \), be a smooth positiv function. If \( f(x, y) = f_1(x) \) for all \( (x, y) \in M \times N \), then \( \tau_f(\varphi) = \tau_{f_1}(\varphi) \), and \( \varphi \) is \( f \)-harmonic map, if and only if, \( \varphi \) is \( f_1 \)-harmonic map (see [5], [10]).

(3) Let \( f_2 : N \to (0, +\infty) \), be a smooth positiv function. If \( f(x, y) = f_2(y) \) for all \( (x, y) \in M \times N \), then \( \tau_f(\varphi) = f_2 \circ \varphi \cdot \tilde{\tau}(\varphi) \), where \( \tilde{\tau}(\varphi) \) denote the tension field of \( \varphi \) between the Riemannian manifolds \((M^m, g)\) and \((N^n, \tilde{h})\) equipped with the conform metric \( \tilde{h} = f_2 \cdot h \).

So \( \varphi : (M^m, g) \to (N^n, h) \) is \( f \)-harmonic map if and only if \( \varphi : (M^m, g) \to (N^n, h) \) is harmonic map.

(4) Let \( f_1 : M \to (0, +\infty) \) and \( f_2 : N \to (0, +\infty) \) be smooth positiv functions.

If \( f(x, y) = f_1(x)f_2(y) \) for all \( (x, y) \in M \times N \), then

\[
\tau_f(\varphi) = f_2 \circ \varphi \{ f_1 \cdot \tilde{\tau}(\varphi) + d\varphi(\text{grad} M f_1) \}
\]

(5) If \( \varphi : (M^m, g) \to (N^n, h) \) has no critical points (i.e. \( |d_x \varphi| \neq 0 \), then harmonic maps, p-harmonic maps and exponential harmonic maps are \( f \)-harmonic map with \( f = 1 \), \( f = |d\varphi|^p \cdot 2 \) and \( f = \exp(|d\varphi|^2) \) respectively.

(6) If \( \varphi : (M^m, g) \to (N^n, h) \) has no critical points, then any F-harmonic map is \( f \)-harmonic map with \( f = F(|d\varphi|^2) \).

Example 1.1. Let \( M = (\mathbb{R}^*, dx^2) \), \( N = (\mathbb{R}, dy^2) \), \( \varphi : M \to N \) be a smooth function, and let \( f : M \times N \to \mathbb{R}_+ \) be a \( C^2 \) function. From Definition 1.2 (formula (1.7)), we have

\[
\tau_f(\varphi) = \left[ f(x, \varphi(x))\varphi''(x) + \frac{\partial f}{\partial x}(x, \varphi(x))\varphi'(x) + \frac{1}{2}\varphi'(x)^2 + \frac{\partial f}{\partial y}(x, \varphi(x)) \right] \frac{d}{dy} \varphi(x).
\]

If \( f(x, y) = e^{x^2} \), by (1.8), \( \varphi \) is harmonic if and only if

\[
\varphi''(x) + \varphi(x)\varphi'(x) + \frac{1}{2}\varphi'(x)^2 = 0.
\]

A local solution of the equation (1.8) is \( \varphi(x) = \frac{x^2}{2} \).

Example 1.2. Let \( \varphi = Id : x \in \mathbb{R}^n \to \varphi(x) = x \in \mathbb{R}^n \), then we have \( \tau(\varphi) = 0 \), \( e(\varphi) = \frac{\partial}{\partial x} \) and from formula (1.7), we obtain :

\[
\tau_f(\varphi) = \left[ \frac{\partial f}{\partial x} + (2 - n) \frac{\partial f}{\partial y} \right] \frac{\partial}{\partial x}.
\]

1.2. The second variation of the \( f \)-energy.

Theorem 1.3. Let \( \varphi : (M^m, g) \to (N^n, h) \) be an \( f \)-harmonic map between Riemannian manifolds, and \( \varphi_{t,s} : M \to N \) (\( -\varepsilon < t, s < \varepsilon \)) be a two-parameter variation with compact support, such that \( \varphi_{0,0} = \varphi \). Set

\[
v = \left. \frac{\partial \varphi_{t,s}}{\partial t} \right|_{t,s=0} \quad \text{and} \quad w = \left. \frac{\partial \varphi_{t,s}}{\partial s} \right|_{t,s=0}.
\]

Under the notation above we have the following:

\[
\left. \frac{\partial^2}{\partial t \partial s} E_f(\varphi_{t,s}) \right|_{t,s=0} = \int_M h(J_{\varphi,f}(v), w)v_g,
\]
where
\[
J_{\varphi,f}(v) = -f \cdot \text{trace}_g R^N (v \circ \varphi) d\varphi - \text{trace}_g \nabla^\varphi f \varphi \nabla^\varphi v
\]
(1.9)
\[
+ e(\varphi)(\nabla^N \text{grad}^N f) \circ \varphi - d\varphi(\text{grad}^M v(f)) - v(f) \tau(\varphi) + \langle \nabla^\varphi v, d\varphi \rangle (\nabla^N \text{grad}^N f) \circ \varphi,
\]
and
\[
\text{trace}_g \nabla^\varphi f \varphi \nabla^\varphi v = \sum_{i=1}^{m} \left( \nabla^\varphi_{e_i} f \varphi \nabla^\varphi_{e_i} v - f \varphi \nabla^\varphi \nabla^N_{e_i} e_i \right).
\]
for any orthonormal frame \((e_i)_i\) on \((M, g)\). Here \(\langle , \rangle\) denote the inner product on \(T^*M \otimes \varphi^{-1}TN\) and \(R^N\) is the curvature tensor on \((N, h)\).

**Definition 1.3.** \(J_{\varphi,f}\) is called the \(f\)-Jacobi operator corresponding to \(\varphi\).

**Proof of Theorem 1.3 :**

Let \(\phi : (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times M \longrightarrow N\) is a map defined by
\[
\phi(t, s, x) = \varphi_{t,s}(x),
\]
where \((-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times M\) is equipped with the product metric. If we extend the vector fields \(\frac{\partial}{\partial t}\) on \((-\varepsilon, \varepsilon)\) and \(\frac{\partial}{\partial s}\) on \((-\varepsilon, \varepsilon)\), then
\[
v = d\phi\left( \frac{\partial}{\partial t} \right)_{t,s=0} \quad \text{and} \quad w = d\phi\left( \frac{\partial}{\partial s} \right)_{t,s=0}.
\]
Let \(\{e_i\}_{i=1}^m\) be an orthonormal frame with respect to \(g\) on \(M\), such that \(\nabla^M e_i = 0\), at fixed point \(x \in M\) for all \(i, j = 1, ..., m\). We compute
\[
\frac{\partial^2}{\partial t \partial s} E_f(\varphi_{t,s}) = \frac{1}{2} \int_M \frac{\partial^2}{\partial t \partial s} \left[ f(x, \varphi_{t,s}(x)) h(d\varphi_{t,s}(e_i), d\varphi_{t,s}(e_i)) \right] v_g,
\]
here summing over the index \(i\). We have
\[
\frac{1}{2} \frac{\partial^2}{\partial t \partial s} \left[ f(x, \varphi_{t,s}(x)) h(d\varphi_{t,s}(e_i), d\varphi_{t,s}(e_i)) \right] =
\]
\[
\frac{1}{2} \frac{\partial^2}{\partial t \partial s} f(x, \varphi_{t,s}(x)) h(d\varphi_{t,s}(e_i), d\varphi_{t,s}(e_i))
\]
(1.10)
\[
+ \frac{\partial}{\partial t} f(x, \varphi_{t,s}(x)) h(\nabla^M_{e_i} d\varphi_{t,s}(e_i), d\varphi_{t,s}(e_i))
\]
\[
+ f(x, \varphi_{t,s}(x)) h(\nabla^M_{e_i} \nabla^M_{e_i} d\varphi_{t,s}(e_i), d\varphi_{t,s}(e_i))
\]
\[
+ f(x, \varphi_{t,s}(x)) h(\nabla^M_{e_i} \nabla^M_{e_i} d\varphi_{t,s}(e_i), \nabla^M_{e_i} d\varphi_{t,s}(e_i)).
\]
Now we calculate each term of right part in the above equation (1.10):
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1. \[
\frac{\partial^2}{\partial t \partial s} f(x, \varphi_{t,s}(x)) = \frac{\partial}{\partial s} \left[ d_{\varphi_{t,s}(x)} I(0, d_{(t,s,x)} \phi(\frac{\partial}{\partial t})) \right] \\
= \frac{\partial}{\partial s} \left[ d_{\varphi_{t,s}(x)} f_{x}(d_{(t,s,x)} \phi(\frac{\partial}{\partial t})) \right] \\
= \frac{\partial}{\partial s} \left[ h(\nabla^N f_{x}, d_{(t,s,x)} \phi(\frac{\partial}{\partial t})) \right] \\
= h(\nabla^\phi \nabla^N f_{x} \circ \phi, d_{(t,s,x)} \phi(\frac{\partial}{\partial t})) \\
+ h(\nabla^N f_{x} \circ \phi, \nabla^\phi d_{(t,s,x)} \phi(\frac{\partial}{\partial t})),
\] then
\[
1.2 \frac{\partial^2}{\partial t \partial s} f(x, \varphi_{t,s}(x)).h(\nabla^\phi d_{\varphi_{t,s}(x)}, d_{\varphi_{t,s}(x)}) \bigg|_{t=s=0} = \\
h(\nabla^N \nabla^N f_{x}(v) e(\varphi) + h \left( (\nabla^N f_{x}(v) \varphi(x), \nabla^\phi d_{\varphi(\frac{\partial}{\partial t})}) \right) \bigg|_{t=s=0} e(\varphi).
\]

2. \[
\frac{\partial}{\partial t} f(x, \varphi_{t,s}(x)).h(\nabla^\phi \phi_{x}, d_{\varphi_{t,s}(x)}, d_{\varphi_{t,s}(x)}) \bigg|_{t=s=0} = \\
h(\nabla^N f_{x} v \phi(\frac{\partial}{\partial s}), d_{\varphi(\frac{\partial}{\partial s})}) \bigg|_{t=s=0} \\
v(f_{x}) \phi(\frac{\partial}{\partial s}), d_{\varphi(\frac{\partial}{\partial s})}) \bigg|_{t=s=0} \\
v(f_{x}) \left( e_{i} (h(w, d_{\varphi(\frac{\partial}{\partial s}))} - h(w, \tau(\varphi)) \right).
\]

If $X$ is the compactly supported vector field on $M$ such for any vector field $Y$ on $M$: \[ g(X, Y) = h(w, d_{\varphi(\phi(Y))} , \] then
\[
\frac{\partial}{\partial t} f(x, \varphi_{t,s}(x)).h(\nabla^\phi \phi_{x}, d_{\varphi_{t,s}(x)}, d_{\varphi_{t,s}(x)}) \bigg|_{t=s=0} = \\
v(f_{x}) \text{div}X - h(w, v(f_{x}) \tau(\varphi)) \\
(1.12) = \text{div}(v(f_{x})X) - h(w, d_{\varphi(\nabla^M v(f_{x}))}) - h(w, v(f_{x}) \tau(\varphi)) \\
(1.12)
\]

3. \[
\frac{\partial}{\partial s} f(x, \varphi_{t,s}(x)).h(\nabla^\phi \phi_{x}, d_{\varphi_{t,s}(x)}, d_{\varphi_{t,s}(x)}) \bigg|_{t=s=0} = \\
h(\nabla^N f_{x}, w), \nabla^\phi v, d_{\varphi} > \\
h(\nabla^\phi v, d_{\varphi} > \nabla^N f_{x}, w),
\]
4. 

\[ f(x, \varphi_{t,s}(x))h(\nabla^\phi_{t,s} \nabla^\psi_{t,s} d\varphi_{t,s}(e_i), d\varphi_{t,s}(e_i)) \bigg|_{t=s=0} \]

\[ = f(x, \varphi(x))h(\nabla^\phi_{t} \nabla^\psi_{t} d\varphi_{t,s}(\frac{\partial}{\partial t}), d\varphi(e_i)) \]

\[ = f_\varphi h(R^N(d\varphi(\frac{\partial}{\partial s}), d\varphi(e_i))d\varphi(\frac{\partial}{\partial t}), d\varphi(e_i)) + f_\varphi h(\nabla^\phi_{e_i} \nabla^\psi_{e_i} d\varphi_{t,s}(\frac{\partial}{\partial t}), d\varphi(e_i)) \]

\[ = -f_\varphi h(R^N(v, d\varphi(e_i))d\varphi(e_i), w) + f_\varphi e_i(h(\nabla^\phi_{e_i} d\varphi_{t,s}(\frac{\partial}{\partial t}), d\varphi(e_i))) \]

\[ (1.14) \]

\[ -f_\varphi h(\nabla^\phi_{t,s} d\varphi_{t,s}(\frac{\partial}{\partial t}), \tau(\varphi)) \]

let \( X_2 \) be a compactly supported vector field on \( M \) such that 

\[ g(X_2, Y) = h(\nabla^\phi_{t,s} d\varphi_{t,s}(\frac{\partial}{\partial t}), d\varphi(Y)) \big|_{t=s=0}, \]

for any vector field \( Y \) on \( M \), then the formula (1.14) becomes

\[ f(x, \varphi_{t,s}(x))h(\nabla^\phi_{t,s} \nabla^\psi_{t,s} d\varphi_{t,s}(e_i), d\varphi_{t,s}(e_i)) \bigg|_{t=s=0} \]

\[ = -f_\varphi h(\text{trace}_g R^N(v, d\varphi)d\varphi, w) \]

\[ + f_\varphi \text{div} X_2 - f_\varphi h(\nabla^\phi_{t,s} d\varphi_{t,s}(\frac{\partial}{\partial t}), \tau(\varphi)) \]

\[ = -f_\varphi h(\text{trace}_g R^N(v, d\varphi)d\varphi, w) + \text{div}(f_\varphi X_2) \]

\[ - h(\nabla^\phi_{e_i} d\varphi_{t,s}(\frac{\partial}{\partial t}), d\varphi(\text{grad}^M f_\varphi)) \]

\[ (1.15) \]

\[ -f_\varphi h(\nabla^\phi_{t,s} d\varphi_{t,s}(\frac{\partial}{\partial t}), \tau(\varphi)) \bigg|_{t=s=0} \]

5. 

\[ f(x, \varphi_{t,s}(x))h(\nabla^\phi_{t,s} d\varphi_{t,s}(e_i), \nabla^\psi_{t,s} d\varphi_{t,s}(e_i)) \bigg|_{t=s=0} \]

\[ = f_\varphi h(\nabla^\phi_{e_i} d\varphi_{t,s}(\frac{\partial}{\partial t}), \nabla^\psi_{e_i} d\varphi_{t,s}(\frac{\partial}{\partial t})) \]

\[ (1.16) \]

Let \( X_3 \) be a compactly supported vector field on \( M \) such that 

\[ g(X_3, Y) = h(\nabla^\phi_{e_i} v, w), \]

for any vector field \( Y \) on \( M \), then the formula (1.16) becomes

\[ f(x, \varphi_{t,s}(x))h(\nabla^\phi_{t,s} d\varphi_{t,s}(e_i), \nabla^\psi_{t,s} d\varphi_{t,s}(e_i)) \bigg|_{t=s=0} \]

\[ = f_\varphi \left[ \text{div} X_3 - h(\text{trace}_g(\nabla^\phi)^2 v, w) \right] \]

\[ = \text{div}(f_\varphi X_3) - h(\nabla^\phi_{\text{grad}^M f_\varphi} v, w) \]

\[ = -h(f_\varphi \text{trace}_g(\nabla^\phi)^2 v, w) \]

\[ (1.17) \]

\[ = \text{div}(f_\varphi X_3) - h(\text{trace}_g(\nabla^\phi)^2 f_\varphi v, w). \]
Substituting the formulas (1.11), (1.12), (1.13), (1.15) and (1.17) in (1.10), and integrate it, the Theorem 1.3 follows.

2. f-Biharmonic Maps

A natural generalization of f-harmonic maps is given by integrating the square of the norm of the f-tension field. More precisely, the f-bi-energy functional of a smooth map \( \varphi : (M^n, g) \longrightarrow (N^n, h) \) is defined by

\[
E_{2,f}(\varphi) = \frac{1}{2} \int_M |\tau_f(\varphi)|^2 v_g.
\]

A map \( \varphi \) is called f-biharmonic if it is a critical point of the f-energy functional.

2.1. First variation of the f-bi-energy.

**Theorem 2.1.** Let \( \varphi : (M^n, g) \longrightarrow (N^n, h) \) be a smooth map and let \( \{\varphi_t\}_t \) \((-\varepsilon < t < \varepsilon)\), be a smooth variation of \( \varphi \). Then

\[
\frac{d}{dt} E_{2,f}(\varphi_t) \bigg|_{t=0} = - \int_M h(\tau_{2,f}(\varphi), v) v_g,
\]

where \( v = \frac{\partial \varphi_t}{\partial t} \bigg|_{t=0} \) denotes the variation vector field of \( \{\varphi_t\}_t \).

\[
\tau_{2,f}(\varphi) = -f(\varphi) \text{trace}_g R^N(\tau_f(\varphi), d\varphi) d\varphi - \text{trace}_g \nabla^\varphi f \nabla^\varphi \tau_f(\varphi) + e(\varphi)(\nabla_{\tau_f(\varphi)} \text{grad}^N f) \circ \varphi - d\varphi(\text{grad}^M \tau_f(\varphi) \circ f) - \tau_f(\varphi) f(\tau_f(\varphi)) + \nabla^\varphi \tau_f(\varphi), d\varphi > (\text{grad}^N f) \circ \varphi.
\]

and

\[
\text{trace}_g \nabla^\varphi f \nabla^\varphi \tau_f(\varphi) = \sum_{i=1}^m \left( \nabla_{e_i}^\varphi f \nabla_{e_i}^\varphi \tau_f(\varphi) - f_{\varphi} \nabla_{e_i}^\varphi \tau_f(\varphi) \right).
\]

for any orthonormal frame \( \{e_i\}_i \) on \((M, g)\)

**Definition 2.1.** \( \tau_{2,f}(\varphi) \) is called the f-bi-tension field of \( \varphi \).

**Proof of Theorem 2.1 :**

Let \( \phi : (-\varepsilon, \varepsilon) \times M \longrightarrow N \) be defined by \( \phi(t, x) = \varphi_t(x) \), where \((-\varepsilon, \varepsilon) \times M \) is equipped with the product metric. We extend the vector fields \(\frac{\partial \phi}{\partial t}\) naturally on \((-\varepsilon, \varepsilon) \times M\), then

\[
\frac{d}{dt} E_{2,f}(\varphi_t) = \int_M h(\nabla_{\frac{\partial \phi}{\partial t}}^\phi \tau_f(\varphi_t), \tau_f(\varphi_t)) v_g,
\]

choose a local orthonormal frame \( \{e_i\}_{1 \leq i \leq m} \) such that \( \nabla_{e_i} e_j = 0 \) for all \( i, j = 1, ..., m \) at a fixed point \( x \in M \), then by (1.7) we have

\[
\nabla_{e_i}^\phi \tau_f(\varphi_t) = \nabla_{e_i}^\phi \nabla_{e_i}^\varphi f \varphi_t(e_i) - \nabla_{e_i}^\phi e(\varphi_t)(\text{grad}^N f) \circ \varphi_t,
\]

First, from formula of curvature tensor, we obtain

\[
\nabla_{e_i}^\phi \nabla_{e_i}^\varphi f \varphi_t(e_i) = R^N(\phi_\varphi(\frac{\partial}{\partial t}), \phi(e_i)) f_{\varphi} d\varphi_t(e_i)
\]

\[
\nabla_{e_i}^\phi \nabla_{e_i}^\varphi f \varphi_t(e_i) = R^N(\phi_{\partial t}(\frac{\partial}{\partial t}), \phi(e_i)) f_{\varphi} d\varphi_t(e_i)
\]

(2.2)
We have
\[ h(\nabla^\phi_{\phi_1} \nabla^\phi_{\phi_1} f, d\varphi_1(e_1), \tau_f(\varphi_i)) = e_i(h(\nabla^\phi_{\phi_1} f, d\varphi_1(e_1), \tau_f(\varphi_i))) \]
\[ -h(\nabla^\phi_{\phi_1} f, d\varphi_1(e_1), \nabla^\phi_{\phi} \tau_f(\varphi_i))) \]
\[ = e_i(h(\nabla^\phi_{\phi_1} f, d\varphi_1(e_1), \tau_f(\varphi_i))) \]
\[ -\partial h(e_i, d\varphi_1(e_1), \nabla^\phi_{\phi} \tau_f(\varphi_i))) \]
\[ -f, h(\nabla^\phi_{\phi_1} d\varphi_1(e_1), \nabla^\phi_{\phi} \tau_f(\varphi_i))). \]

let \( X \) be the compactly supported vector field on \( M \) such that
\[ g(X, Y) = h(\nabla^\phi_{\phi} f, d\varphi_1(Y), \tau_f(\varphi_i)) \bigg|_{t=0}, \quad \forall Y \in \Gamma(TM), \]
then
\[ h(\nabla^\phi_{\phi_1} \nabla^\phi_{\phi_1} f, d\varphi_1(e_1), \tau_f(\varphi_i)) \bigg|_{t=0} = \]
\[ \text{div} X - h(\text{grad} f, v), < d\varphi, \nabla^\phi \tau_f(\varphi) > \]
\[ -f, h(\nabla^\phi_{\phi_1} d\varphi_1(e_1), \nabla^\phi_{\phi} \tau_f(\varphi))) \bigg|_{t=0} \]
\[ = \text{div} X - h(\text{grad} f, v), < d\varphi, \nabla^\phi \tau_f(\varphi) > \]
\[ -f, e_i( h(v, \nabla^\phi \tau_f(\varphi))) - h(v, \nabla^\phi_{\phi} \tau_f(\varphi))) \bigg| \]
\[ = \text{div} X - h(\text{grad} f, v), < d\varphi, \nabla^\phi \tau_f(\varphi) > \]
\[ -f, e_i \left( \text{div} X_2 - h(v, \text{trace}_g(\nabla^\phi)^2 \tau_f(\varphi)) \right) \]
\[ = \text{div} X - h(\text{grad} f, v), < d\varphi, \nabla^\phi \tau_f(\varphi) > \]
\[ -f, e_i \left( \text{div} X_2 - h(v, \text{trace}_g(\nabla^\phi)^2 \tau_f(\varphi)) \right) \]
\[ \bigg(2.3\bigg) \]

where \( X_2 \) is the compactly supported vector field on \( M \) such that
\[ g(X_2, Y) = h(v, \nabla^\phi \tau_f(\varphi)), \quad \forall Y \in \Gamma(TM), \]
By the formulas (2.2) and (2.3), we have
\[ h(\nabla^\phi_{\phi_1} \nabla^\phi_{\phi_1} f, d\varphi_1(e_1), \tau_f(\varphi_i)) \bigg|_{t=0} = \]
\[ h(f, \text{trace}_g R^N(\tau_f(\varphi), d\varphi), d\varphi) \]
\[ + \text{div} X - h(< d\varphi, \nabla^\phi \tau_f(\varphi) > . \text{grad} f, v) \]
\[ -\text{div}(f, X_2) + h(v, \nabla^\phi_{\text{grad}^N f} \tau_f(\varphi)) \]
\[ + h(v, f, \text{trace}_g(\nabla^\phi)^2 \tau_f(\varphi)) \]
\[ = h(f, \text{trace}_g R^N(\tau_f(\varphi), d\varphi), d\varphi) \]
\[ + \text{div} X - h(< d\varphi, \nabla^\phi \tau_f(\varphi) > . \text{grad} f, v) \]
\[ -\text{div}(f, X_2) + h(f, \text{trace}_g(\nabla^\phi)^2 \tau_f(\varphi), v). \]
\[ \bigg(2.4\bigg) \]
On the other hand, we have
\[ \bigg(2.5\bigg) \]
\[ \nabla^\phi_{\phi} e(\varphi_i)(\text{grad} f) \circ \phi = \frac{\partial e(\varphi_i)}{\partial t}(\text{grad} f) \circ \phi + e(\varphi_i) \nabla^\phi_{\phi} (\text{grad} f) \circ \phi, \]
since
\[
\frac{\partial \phi(\varphi_t)}{\partial t} \bigg|_{t=0} = h(\nabla^\phi_{\varphi_t} \phi, d\phi(\varphi_t)) \bigg|_{t=0} = h(\nabla^\phi_{\varphi_t} \phi, d\phi(\varphi_t)) \bigg|_{t=0}
\]
\[
= h(\nabla^\phi_{\varphi_t} \phi, d\phi(\varphi_t)) \bigg|_{t=0}
\]
\[
= e_t(h(v, d\phi(e_t))) - h(v, \tau(\varphi)).
\]
\[(2.6)
\]
where \(X_3\) is the compactly supported vector field on \(M\) such that
\[
g(X_3, Y) = h(v, d\phi(Y)), \quad \forall Y \in \Gamma(TM).
\]

By the formulas (2.5) and (2.6), we obtain
\[
\begin{align*}
&h(\nabla^\phi_{\varphi_t} \phi, (\text{grad}^M f) \circ \phi, \tau(\varphi)) \bigg|_{t=0} \\
&= \tau(\varphi)(\varphi) \circ \phi, \tau(\varphi) \bigg|_{t=0} \\
&= h(\nabla^\phi_{\varphi_t} \phi, (\text{grad}^M f) \circ \phi, \tau(\varphi)) \\
&= \text{div}(\tau(\varphi)(\varphi) \circ \phi) - h(v, d\phi(\tau(\varphi)(\varphi))) \\
&= \text{div}(\tau(\varphi)(\varphi) \circ \phi) + e(\phi)h(\nabla^\phi_{\varphi_t} \phi, \text{grad}^M f, v)
\end{align*}
\]
\[(2.7)
\]
Substituting the formulas (2.4) and (2.7) in (2.1), the Theorem 2.1 follows.

From Theorem 2.1, we deduce

**Theorem 2.2.** Let \(\varphi : (M^m, g) \rightarrow (N^n, h)\) be a smooth map. Then \(\varphi\) is \(f\)-biharmonic if and only if we have:

\[
\tau_{2.f}(\varphi) = -f \text{trace} \nabla^N \tau_f(\varphi), d\phi - \text{trace} \nabla^\varphi \tau_f(\varphi) + e(\phi)(\text{grad}^M \tau_f(\varphi)(\varphi)) - \tau_f(\varphi)(\varphi) \circ \phi - d\phi(\text{grad}^M \tau_f(\varphi)(\varphi))< \nabla^\varphi \tau_f(\varphi), d\phi > (\text{grad}^M f) \circ \phi.
\]

\[
= 0
\]

**Particular Cases**

1. If \(f = 1\), then \(\tau_{2.f}(\varphi) = \tau(\varphi)\), is the natural bi-tension field of \(\varphi\) (see [11], [12]).
2. Let \(f_1 : M \rightarrow (0, +\infty)\), be a smooth positive function. If \(f(x, y) = f_1(x)\) for all \((x, y) \in M \times N\), then \(\tau_{2.f}(\varphi) = \tau_{2.f_1}(\varphi)\), and \(\varphi\) is \(f_1\)-biharmonic map, if and only if, \(\varphi\) is \(f_1\)-biharmonic map (see [10]).
3. Let \(f_2 : N \rightarrow (0, +\infty)\), be a smooth positive function. If \(f(x, y) = f_2(y)\) for all \((x, y) \in M \times N\), then \(\varphi : (M^m, g) \rightarrow (N^n, h)\) is \(f_2\)-biharmonic map if and only if \(\varphi : (M^m, g) \rightarrow (N^n, h)\) is bi-harmonic map, where \((N^n, h)\) equipped with the conform metric \(\tilde{h} = f_2 h\) (see [11]).

**References**


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