INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 12 NO. 1 PAGE 116–125 (2019)

Some Notes Concerning Riemannian Submersions and Riemannian Homogenous Spaces

Mehmet Gülbahar*, Erol Kılıç and Sadık Keleş

(Communicated by Murat Tosun)

ABSTRACT

Riemannian submersions between Lie groups and Riemannian homogeneous spaces are investigated. With the help of connections, some characterizations dealing these spaces are obtained.

Keywords: *Riemann submersion; Lie group; Riemannian homogeneous space; connection.*

AMS Subject Classification (2010): *Primary: 22F30; Secondary: 53C25; 53C30.*

1. Introduction

According to F. Klein, the main purpose of studying geometry is to investigate invariant properties of geometrical shapes and figures under the actions of special group transformations. The groups which lead to determine various geometries, known as Lie groups in the literature, are considered and developed by S. Lie.

A smooth manifold possesses a smooth group structure is called a Lie group. The simple and best known examples of Lie groups are the groups of isometries of the real Euclidean space E^n, the complex Euclidean space C^n, the quaternion space H^n. Hence, these groups of isometries formed to rise of commonly used groups such as the general linear groups $GL(n, \mathrm{R})$ and $GL(n, \mathrm{C})$, the orthogonal group $\mathrm{O}(n)$, the unitary group $\mathrm{U}(n)$, the symplectic group $\mathrm{Sp}(n)$ etc.

Another important fact dealing Lie groups is to investigate the differential geometry of Lie algebras $\mathfrak{g}$ which are corresponding to tangent spaces of Lie groups. With the aid of smooth maps (such as immersions or submersion etc.) between Lie algebras of any two Lie groups, some basic geometrical and algebraic properties any two Lie algebras or Lie groups could be investigated and some basic relationships between Lie groups could be proved. In this sense, there exist various papers analyzed the notions immersions (cf. [7, 8, 11, 12, 19]) and submersions (cf. [3, 4, 9, 16, 17, 20]) on Lie groups.

Motivated by this facts, we shall present some relations between a Lie group and a reductive Riemannian homogeneous space whose tangent spaces always admits a submersion.

2. Riemannian Submersions

Let (M, g) and (B, g') be m and n dimensionals Riemannian manifolds with Riemannian metrics g and $\widetilde{g}$, respectively. A smooth map $\pi : (M, g) \to (B, \widetilde{g})$ is called a *Riemannian submersion* if

i) π has maximal rank.

Received : *01-November-2018,* ***Accepted :*** *01-February-2019*
**** Corresponding author***

This article is the written version of author's plenary talk delivered on August 28-31, 2018 at 7th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2018) at Kyiv-Ukraine

ii) The differential $d\pi$ preserves the lengths of horizontal vectors.

Let $\pi : (M, g) \to (B, g')$ be a Riemannian submersion. For any $b \in B$, $\pi^{-1}(b)$ becomes a closed r-dimensional submanifold of M. The submanifolds $\pi^{-1}(b)$ are called as *fibers*. A vector field tangent to fibers is called *vertical* and a vector field orthogonal to fibers is called *horizontal*. If we put

$$\mathcal{V}_p = \text{kernel}(\pi_*) \tag{2.1}$$

at a point $p \in M$, then it can be obtained an integrable distribution $\mathcal{V}$ corresponding to the foliation of M determined by the fibres of π. The distribution $\mathcal{V}_p$ is called *vertical space* at $p \in M$. Sections of $\mathcal{V}$ are so-called *vertical vector fields* and the set of all vertical vector fields is denoted by $\chi^v(M)$.

Let $\mathcal{H}$ be the complementary distribution of $\mathcal{V}$ determined by the Riemannian metric g. Then, we always have the following orthogonal decomposition for a Riemannian submersion:

$$TM = \mathcal{V} \oplus \mathcal{H}. \tag{2.2}$$

For any $p \in M$, the distribution $\mathcal{H}_p = (\mathcal{V}_p)^{\perp}$ is called *horizontal space* on M [13]. Furthermore, every section of $\mathcal{H}$ is so-called *horizontal vector field* and all horizontal vector fields set up a subspace $\chi^h(M)$ in $\chi(M)$.

A vector field E on M is called *basic* if it is horizontal and $\pi-$ related to a vector field E' on B i.e., $\pi_* E_p = E'_{\pi(p)}$ for all $p \in M$. The space of all π-related vector fields on B is denoted by $\chi^b(M)$. If $E, F \in \chi^b(M)$ are $\pi-$related to E_* and F_* respectively, then one has

$$g(E, F) = g'(E', F') \circ \pi. \tag{2.3}$$

Now we recall the following proposition of [6]:

Proposition 2.1. *Let $\pi : (M, g) \to (B, g')$ be a Riemannian submersion. Denote ∇ and ∇' to be the Levi-Civita connections of M and B, respectively. If X and Y are basic vector fields, π-related to X' and Y', respectively, one has*

i) $g(X, Y) = g'(X', Y') \circ \pi$;

ii) $h[X, Y]$ *is the basic vector field π-related to* $[X', Y']$;

iii) $h\nabla_X Y$ *is the basic vector field π-related to* $\nabla'_{X'} Y'$;

iv) $[X, V]$ *is vertical for any vertical vector field* V.

Suppose h and v are the projections of $\chi(M)$ onto $\chi^h(M)$ and $\chi^v(M)$ respectively. The *fundamental tensor fields* A and T of π, are defined to be

$$A_E F = h\nabla_{hE} vF + v\nabla_{hE} hF, \tag{2.4}$$
$$T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF \tag{2.5}$$

for any $E, F \in \chi(M)$.

Let us define the following mappings [10]:

$$\begin{aligned} T^{\mathcal{H}} : \chi^v(M) \times \chi^v(M) &\to \chi^h(M), \\ (U, V) &\to T^{\mathcal{H}}(U, V) = h\nabla_U V, \end{aligned} \tag{2.6}$$

$$\begin{aligned} T^{\mathcal{V}} : \chi^v(M) \times \chi^h(M) &\to \chi^v(M), \\ (U, X) &\to T^{\mathcal{V}}(U, X) = v\nabla_U X, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} A^{\mathcal{H}} : \chi^h(M) \times \chi^v(M) &\to \chi^h(M), \\ (X, U) &\to A^{\mathcal{H}}(X, U) = h\nabla_X U, \end{aligned} \tag{2.8}$$

$$\begin{aligned} A^{\mathcal{V}} : \chi^h(M) \times \chi^h(M) &\rightarrow \chi^v(M), \\ (X,Y) &\rightarrow A^{\mathcal{V}}(X,Y) = v\nabla_X Y, \end{aligned} \tag{2.9}$$

Then, it is clear from (2.4), (2.5), (2.6) and (2.9) that $T^{\mathcal{H}}$ is a symmetric operator on $\chi^v(M) \times \chi^v(M)$ and $A^{\mathcal{V}}$ is an anti-symmetric operator on $\chi^h(M) \times \chi^h(M)$. Also, if we take into account (2.4) and (2.5) in (2.2), we can write

$$\nabla_U V = T^{\mathcal{H}}(U,V) + v\nabla_U V, \tag{2.10}$$
$$\nabla_V X = h\nabla_V X + T^{\mathcal{V}}(U,X), \tag{2.11}$$
$$\nabla_X U = A^{\mathcal{H}}(X,U) + v\nabla_X U, \tag{2.12}$$
$$\nabla_X Y = h\nabla_X Y + A^{\mathcal{V}}(X,Y) \tag{2.13}$$

for any $U, V \in \chi^v(M)$ and $X, Y \in \chi^h(M)$.

Now we recall the following theorem [6]:

Theorem 2.1. *Let $\pi : (M,g) \rightarrow (B,g')$ be a Riemannian submersion. Then the horizontal space $\mathcal{H}$ becomes an integrable distribution if and only if the fundamental tensor A vanishes identically.*

Remark 2.1. As a consequence of Theorem 2.1, we see that both $A^{\mathcal{H}}$ and $A^{\mathcal{V}}$ are related to integrability of $\mathcal{H}$, that is, they are identically zero if and only if $\mathcal{H}$ is integrable.

Let R, R' and $\hat{R}$ be the curvature tensors on M, B and fibers $\pi^{-1}(b)$ respectively, and $\check{R}(X,Y)Z$ denotes the horizontal lift of $R'_{\pi(b)}(\pi_{*p}X_b, \pi_{*p}Y_b)Z_b$ at any point $b \in M$ satisfying

$$\pi_*(\check{R}(X,Y)Z) = R'(\pi_* X, \pi_* Y)\pi_* Z.$$

Then, the following relations between these tensors hold [18]:

$$\begin{aligned} R(U,V,W,G) &= \hat{R}(U,V,W,G) + g\left((T^{\mathcal{H}}(U,G), T^{\mathcal{H}}(V,W)\right) \\ &\quad -g\left(T^{\mathcal{H}}(V,G), T^{\mathcal{H}}(U,W)\right), \end{aligned} \tag{2.14}$$

$$\begin{aligned} R(X,Y,Z,H) &= \check{R}(X,Y,Z,H) - 2g\left(A^{\mathcal{V}}(X,Y), A^{\mathcal{V}}(Z,H)\right) \\ &\quad +g\left(A^{\mathcal{V}}(Y,Z), A^{\mathcal{V}}(X,H)\right) \\ &\quad -g\left(A^{\mathcal{V}}(X,Z), A^{\mathcal{V}}(Y,H)\right), \end{aligned} \tag{2.15}$$

$$\begin{aligned} R(X,V,Y,W) &= g\left(\left(\nabla_X T\right)(V,W), Y\right) + g\left(\left(\nabla_V A\right)(X,Y), W\right) \\ &\quad -g\left(T^{\mathcal{V}}(V,X), T^{\mathcal{V}}(W,Y)\right) \\ &\quad +g\left(A^{\mathcal{H}}(X,V), A^{\mathcal{H}}(Y,W)\right), \end{aligned} \tag{2.16}$$

for any $U, V, W, G \in \chi^v(M)$ and $X, Y, Z, H \in \chi^h(M)$. With the help of (2.14) $-$ (2.16) equations, we get

$$K(U,V) = \hat{K}(U,V) - \left\|T^{\mathcal{H}}(U,V)\right\|^2 + g\left(T^{\mathcal{H}}(U,U), T^{\mathcal{H}}(V,V)\right), \tag{2.17}$$
$$K(X,Y) = \check{K}(\check{X},\check{Y}) + 3\|A^{\mathcal{V}}(X,Y)\|^2, \tag{2.18}$$
$$\begin{aligned} K(X,V) &= -g\left(\left(\nabla_X T\right)(V,V), X\right) + \left\|T^{\mathcal{V}}(V,X)\right\|^2 \\ &\quad - \left\|A^{\mathcal{H}}(X,V)\right\|^2, \end{aligned} \tag{2.19}$$

where K, $\hat{K}$ and $\check{K}$ denote the sectional curvatures in M, any fiber $\pi^{-1}(b)$ and the horizontal distribution $\mathcal{H}$ respectively.

3. Lie Groups and Algebras

Let G be a Lie group and V be a set. A mapping $\phi : G \times V \rightarrow V$ is called an action of G on V if, for all $g, h \in G$ and $v \in V$, the following relations hold:

i) $e \cdot v = v$.

ii) $g \cdot (h \cdot v) = gh \cdot v$,

where $\phi(g, v) = g \cdot v$. Furthermore,

a) The set $G_x = \{g \in G : g \cdot x = x\}$ is called the isotropy group (subgroup) at $x \in G$.

b) The set $G \cdot x = \{g \cdot x : g \in G\}$ is called orbit at a point $x \in G$.

A Lie group G acts on itself by the left and the right translations. Another important action G is the adjoint action that a homomorphism $I_g : G \to G$ defined by

$$I_g(a) = gag^{-1} \text{ for all } a \in G. \tag{3.1}$$

We note that the adjoint action is a group homomorphism on G whereas the right and the left translations are not. Therefore, any curve through the identity element e of G mapped by the adjoint map to another (not necessarily the same) curve through e since $I_g(e) = e$.

A vector field X is called left invariant if

$$dL_a(X_e) = X_a, \tag{3.2}$$

where e is the identity element of G. Here, L_a denotes a left translation on $a \in G$.

In a similar manner, a vector field X is called right invariant if

$$dR_a(X_e) = X_a. \tag{3.3}$$

Here, R_a denotes right translation on $a \in G$.

Let $\mathfrak{g}$ be the set of all left (or right) invariant vector fields on G. We note that $\mathfrak{g}$ is a vector space with the usual addition of vector fields, and there exists a linear isomorphism between $\mathfrak{g}$ and the tangent space T_eG. Furthermore, $\mathfrak{g}$ is closed under the bracket operation and it is called the Lie algebra of G.

The exponential map $\exp : \mathfrak{g} \to G$ is defined by

$$\exp(X) = \varphi_X(t), \ \forall t \in \mathrm{R} \tag{3.4}$$

where $\varphi_X(t)$ denotes flows (one parameter transformation groups) generated by a vector field X in $\mathfrak{g}$ satisfying $\varphi_X(0) = e$ and $\varphi'_X(0) = X_e$.

Now, let K be a closed subgroup of G and $\mathfrak{k}$ be its Lie algebra defined by

$$\mathfrak{k} = \{X \in \mathfrak{g} : \exp(tX) \in \mathfrak{h}, \text{ for all } t \in \mathbb{R}\}. \tag{3.5}$$

It is known from the Cartan's subgroup theorem that K is a submanifold of G which is also called as Lie subgroup of G [7].

Furthermore, the adjoint representation Adj : $G \to \mathrm{Aut}(\mathfrak{g})$ is defined by

$$\mathrm{Ad}(g) = dI_g, \tag{3.6}$$

where

$$\mathrm{Ad}(g)X = dI_g = d(R_{g^{-1}} \circ L_g) \tag{3.7}$$

for all $X \in \mathfrak{g}$ [2]. Here $\mathrm{Aut}(\mathfrak{g})$ denotes the set of all automorphisms on $\mathfrak{g}$. Therefore the adjoint representation maps any vector (of a curve on G) in $\mathfrak{g}$ to another vector in $\mathfrak{g}$. In contrast dL_g and dR_g map tangent vectors in T_eG to tangent vectors in T_gG.

A Riemannian metric $\langle , \rangle$ on a Lie group G is called left invariant if

$$\langle X, Y \rangle = \langle dL_aX, dL_aY \rangle \tag{3.8}$$

for all $a \in G$ and $X, Y \in \mathfrak{g}$. The left invariant metrics clearly have property that the translations L_a are isometries for all $a \in G$. Similarly, a Riemannian metric is called right invariant if each right translations on G is a isometry.

A metric on G that is both left invariant and right invariant is called bi-invariant. We note that a compact Lie group always admits a bi-invariant metric [2].

Proposition 3.1. *[2] Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Then, there is one-to-one correspondence between bi-invariant metric on G and Ad-invariant scalar products on $\mathfrak{g}$ i.e.,*

$$\langle Ad(g)X, Ad(g)Y\rangle = \langle X, Y\rangle \tag{3.9}$$

for all $g \in G$ and $X, Y \in \mathfrak{g}$.

Proposition 3.2. *[2] Let G be a Lie group with a bi-invariant metric $\langle , \rangle$. Then for all $X, Y, Z \in \mathfrak{g}$:*

i) The Riemann connection is given by

$$\nabla_X Y = \frac{1}{2}[X, Y]. \tag{3.10}$$

ii) The Riemann curvature tensor is given by

$$R(X, Y)Z = \frac{1}{4}[[X, Y], Z]. \tag{3.11}$$

ii) The sectional curvature of a plane section Π spanned by X and Y is given by

$$K(\Pi) = \frac{1}{4}\frac{\langle [X, Y], [X, Y]\rangle}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2}. \tag{3.12}$$

4. Riemannian Homogeneous Spaces

Let G be a Lie group and K be a closed subgroup of G. Consider the following quotient space

$$G/K = \{gK : g \in G\}. \tag{4.1}$$

Then, it is said to be that G acts transitively in a natural way on G/K if, for every pair of elements x and y, there exists a element g in G such that $g \cdot x = y$.

A quotient space G/K acted by G transitively is called a homogeneous space [2].

Let G/K be a homogeneous space and $\pi : G \to G/K$ be a projection such that

$$\begin{aligned} \pi \quad : \quad & G \to G/K \\ & g \to \pi(g) = gK. \end{aligned}$$

For each $a \in G$, let τ_a be the left translation on G/K that sends each gK to agK. If L_a is a left translation on G then we have the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\ \pi\ } & G/K \\ {\scriptstyle L_a}\downarrow & & \downarrow{\scriptstyle \tau_a} \\ G & \xrightarrow{\ \pi\ } & G/K \end{array}$$

Thus, we get

$$\pi \circ L_a = \tau_a \circ \pi. \tag{4.2}$$

Let G/K be a homogeneous space and $\pi : G \to G/K$ be a projection. Now, we shall investigate the differential

$$d\pi_e : \mathfrak{g} \to T_o(G/K),$$

where $o = \pi(e) = K$. We note that any quotient space G/K acted by G is a manifold iff $\pi : G \to G/K$ is a submersion [2].

Now, let $X \in \mathfrak{g}$ and the exponential map $\exp tX$ be the corresponding one-parameter subgroup. Then

$$d\pi_e(X) = \frac{d}{dt}(\pi \circ \exp tX)|_{t=0} = \frac{d}{dt}((\exp tX)K)|_{t=0}. \tag{4.3}$$

Denote the vertical distribution of $\mathfrak{g}$ by $\mathcal{V}_e$. Then, there exists the following canonical isomorphism:

$$\mathfrak{g}/\mathcal{V}_e \cong T_o(G/K) \tag{4.4}$$

A homogeneous space is called 'reductive' if there exists a subspace $\mathcal{H}_e$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathcal{V}_e \oplus \mathcal{H}_e \tag{4.5}$$

and $\mathcal{H}_e$ is invariant under adjoint representation of K. In this case, we have

$$\mathcal{H}_e \cong T_o(G/K). \tag{4.6}$$

Here, $\mathcal{H}$ becomes the horizontal distribution of G with respect to π.

Example 4.1. Let us consider the special orthogonal group

$$SO(n+1) = \{A \in M(n+1) : A^tA = I_{n+1} \text{ and } \det A = 1\}$$

Here, $M(n+1)$ denotes the set of all $(n+1)\times(n+1)$ matrices, I_{n+1} is the identity matrix and A^t is the transpose of A.

The group $SO(n+1)$ acts transitively on the unit sphere S^n:

In fact, let $x, y \in S^n$ and $\{x, a_1, a_2, \dots a_n\}$ and $\{y, b_1, b_2, \dots b_n\}$ are two orthonormal basis of $\mathbb{R}^{n+1}$ with the same orientation. The translation matrix lies in $SO(n+1)$. Furthermore, isotropy subgroup of at the point $(1, 0, \dots, 0)$ consists of the matrices in form

$$\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} \in SO(n+1),$$

where $A \in SO(n)$. This isotropy group is denoted by $SO(n)$ and hence we have $SO(n+1)/SO(n) = S^n$ for $n \neq 1$.

For $n = 1$, consider the special orthogonal group $SO(2)$ given by

$$SO(2) = \{\begin{bmatrix} a & b \\ -b & a \end{bmatrix} : \quad a^2 + b^2 = 1, \quad a, b \in \mathbb{R}\}.$$

Then a map on $SO(2)$ to unit sphere S^1 defined by

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \to z = a + ib = (a, b),$$

implies that $SO(2) \cong S^1$. Here we note that the unit sphere is a Lie group under multiplication in complex number space $\mathbb{C}$.

For $n = 2$, we see that $SO(3)/SO(2) = S^2$. But S^2 does not admit a topological group structure. Thus, S^2 isn't a Lie group [14].

For $n = 3$, we see that $SO(4)/SO(3) = S^3$. We note that S^3 is a Lie group under multiplication in quaternion space $\mathbb{H}$.

Now, we shall show that $\pi : SO(4) \to S^3$ is a Riemannian submersion.

Since the Lie algebra of $SO(4)$ is the set of all 4×4 anti-symmetric matrices denoted by $so(4)$, the differential $d\pi|_e$ is a surjective mapping on $so(4)$ to T_oS^3, where $e = I_4$ and $\pi(e) = o = (1, 0, 0, 0)$.

Here, for any

$$X = \begin{bmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & 0 & x_4 & x_5 \\ -x_2 & -x_4 & 0 & x_6 \\ -x_3 & -x_5 & -x_6 & 0 \end{bmatrix} \in so(4),$$

the deferential of π is defined by

$$d\pi(X) = (0, x_1, x_2, x_3).$$

In this case,

$$\mathcal{V}_e = so(3),$$

where $So(3)$ is a subspace of $so(4)$ formed as

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} : A \in so(3) \right\}.$$

Furthermore, the horizontal space $\mathcal{H}_e$ at $e \in SO(4)$ becomes

$$\mathcal{H}_e = \left\{ \begin{bmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & & & \\ -x_2 & & 0_3 & \\ -x_3 & & & \end{bmatrix} : x_1, x_2, x_3 \in \mathrm{R} \right\},$$

where 0_3 denotes 3×3 zero matrix.

For any two horizontal vector fields given by

$$X = \begin{bmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & & & \\ -x_2 & & 0_3 & \\ -x_3 & & & \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 & y_1 & y_2 & y_3 \\ -y_1 & & & \\ -y_2 & & 0_3 & \\ -y_3 & & & \end{bmatrix}$$

we have

$$\begin{aligned} \langle X, Y \rangle &= \frac{1}{4}\text{trace}\left(X^t Y\right) \\ &= x_1^2 + x_2^2 + x_3^2 \\ &= \langle d\pi(X), d\pi(Y) \rangle. \end{aligned}$$

Therefore, π is a Riemannian submersion.

Example 4.2. Consider the 3-dimensional Euclidean space E^3 and the 1-dimensional Euclidean space E. Let us define the following projection:

$$\begin{aligned} \pi : \mathrm{E}^3 &\to \mathrm{E} \\ (x_1, x_2, x_3) &\to \pi(x_1, x_2, x_3) = x_1. \end{aligned}$$

Then, it is clear that π is a surjective homomorphism. Thus, we have

$$\ker \pi = \{(0, x_2, x_3) : x_2, x_3 \in \mathrm{R}\} \cong \mathrm{E}^2$$

and

$$\mathrm{E}^3/\mathrm{E}^2 \cong \mathrm{E}.$$

By a straightforward computation, one can see that $d\pi$ is an local isometry and π is a Riemannian submersion.

Further examples of Riemannian homogeneous spaces admitting a Riemannian submersions could be given in projective spaces, quaternion spaces, Grassmann manifolds, Steifel manifolds, flag manifolds etc.

Let G/K be a reductive Riemannian homogeneous spacec. A metric $\langle,\rangle$ on G/K is called G-invariant if, for each $a \in G$ and $X, Y \in \mathcal{H}_e$, the following relation holds [2]:

$$\langle X, Y\rangle = \langle d\tau_a(X), d\tau_a(Y)\rangle. \tag{4.7}$$

Theorem 4.1. *Let G/K be a Riemannian homogeneous space. If G/K posseses a G-invariant metric, then the mapping π is a Riemannian submersion.*

Proof. Let G/K be a Riemannian homogeneous space with a G invariant metric $\langle,\rangle$. it is clear that π is submersion since π is a covering map. Now we shall show that π is a Riemannian submersion.

From the definition of pushforward map, we have

$$d(\tau_a \circ \pi)X = X(\tau_a \circ \pi), \quad \forall a \in G \tag{4.8}$$

for any horizontal vector field X. Using the fact that $\tau_a \circ \pi = \pi \circ L_a$ (see Figure 1), it follows that

$$\begin{aligned} d(\tau_a \circ \pi)X &= X(\pi \circ L_a) \\ &= d(\pi \circ L_a)X \end{aligned} \tag{4.9}$$

Next, from (4.7) and (4.9), we obtain

$$\langle d(\pi \circ L_a)X, d(\pi \circ L_a)Y\rangle = \langle X, Y\rangle \tag{4.10}$$

for any horizontal vector fields X and Y. The last equation implies that π is a Riemannian submersion. □

Let G/K be a reductive Riemannian homogeneous space with a G invariant metric $\langle,\rangle$ and the Riemannian connection ∇^*. Let $X \in \mathfrak{g}$ and the exponential map $\exp(tX)$ be the corresponding one parameter subgroup (a flow) generated by X. In this case, we write

$$d\pi(X) = X_o^* = \frac{d}{dt}(\exp tX) \cdot o|_{t=0}, \tag{4.11}$$

Here, we note that X^* is a Killing field in T_oG/K. Since the mapping π is a canonical projection, we may write $d\pi(X) = X_o^*$ and $d\pi(hX) = X_o^*$, where hX is the horizontal component of $X \in \mathfrak{g}$ with respect to π. Also, since X^* is a Killing vector field, its flows are isometries and the following equations hold:

$$X^*\langle Y, Z\rangle = \langle [X^*, Y]_{G/K}, Z\rangle + \langle Y, [X^*, Z]_{G/K}\rangle, \tag{4.12}$$

$$\langle \nabla^*_Y X^*, Z\rangle + \langle \nabla^*_Z X^*, Y\rangle = 0 \tag{4.13}$$

and

$$[X^*, Y^*]_{G/K} = -[X, Y]^* \tag{4.14}$$

for all $Y, Z \in \chi(G/K)$. For more details, we refer to [1], [5] and [15].

Let $X, Y \in \mathcal{H}_e$. Then

$$(\nabla^*_{X^*} Y^*)_o = -\frac{1}{2}h[X, Y] + U(X, Y) \tag{4.15}$$

where $U : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is determined by

$$2g(U(X, Y), Z) = g(h[Z, X], Y) + g(X, h[Z, Y]) \tag{4.16}$$

for all $Z \in \mathcal{H}_e$.

The manifold G/K is called naturally reductive Riemannian homogeneous space if $U = 0$.

Proposition 4.1. *Let G/K be a Riemannian homogeneous space. If X^* and Y^* are Killing vector fields on T_oG/K such that $\pi(e) = o$. Then we have*

$$\nabla^*_{X^*} Y^* = -\frac{1}{2}[X,Y]^* + U(X,Y) \tag{4.17}$$

for any $X, Y \in \mathcal{H}_e$.

Proof. Let X and Y are any horizontal (or basic) vector fields in $\mathfrak{g}$ related X^* and Y^* in $T_o(G/K)$, that is,

$$d\pi(X) = X^* \text{ and } d\pi(Y) = Y^*. \tag{4.18}$$

Using (4.14), we get

$$d\pi(h[X,Y]) = -[X^*, Y^*]_{G/K}. \tag{4.19}$$

From (4.15) and (4.19), the proof of proposition is straightforward. □

Taking into account of Proposition 2.1 and Proposition 4.1, we get the following:

Corollary 4.1. *Let G/K be a Riemannian homogeneous space. If $\pi : G \to G/K$ is a Riemannian submersion, then there exists a Lie algebra isomorphism between $\mathcal{H}_e$ and $T_o(G/K)$.*

Proposition 4.2. *Let G/K is a naturally reductive Riemannian homogeneous space. For any vector fields X, Y and Z in $\mathcal{H}_e$ which are π related to X^*, Y^* and Z^*, respectively, we have*

i) $(\nabla^*_{X^*} Y^*)_o = -\frac{1}{2}[X,Y]^*$,
ii) $R^*(X^*,Y^*)Z^* = \frac{1}{4}[Y,[X,Z]]^* - \frac{1}{4}[X,[Y,Z]]^* + \frac{1}{2}[[X,Y],Z]^*$.
iii) If $\mathcal{H}_e$ is integrable, then $R^*(X^*,Y^*)Z^* = -\frac{3}{4}[X,Y],Z]^*$.

Here, R^* denotes the Riemannian curvature tensor on G/K.

Proof. Using $U = 0$ in (4.15) and taking into consideration to Proposition 4.1, we find (i).

From the statement (i), we get

$$\begin{aligned} R^*(X^*,Y^*)Z^* &= \nabla^*_{Y^*}\nabla^*_{X^*}Z^* - \nabla^*_{X^*}\nabla^*_{Y^*}Z^* + \nabla^*_{[X,Y]^*}Z^* \\ &= \nabla^*_{Y^*}\left(-\frac{1}{2}[X,Z]^*\right) - \nabla^*_{X^*}\left(-\frac{1}{2}[Y,Z]^*\right) - \frac{1}{2}[[X,Y],Z]^* \\ &= \frac{1}{4}[Y,[X,Z]]^* - \frac{1}{4}[X,[Y,Z]]^* - \frac{1}{2}[[X,Y],Z]^*. \end{aligned} \tag{4.20}$$

which completes the proof of (ii).

Taking into account of Corollary 4.1 and using the Jakobi property for Lie brackets in (4.20), the proof of (iii) is straightforward. □

From (3.11) and the statement (iii) of Corollary 4.2, we get the following corollary:

Corollary 4.2. *Let G be a Lie group with a bi-invariant metric and G/K be a naturally reductive Riemannian homogeneous space. If $\mathcal{H}_e$ is an integrable distribution, then we have*

$$\pi_*(R(X,Y)Z) = -3R^*(X^*,Y^*)Z^* \tag{4.21}$$

for any X, Y and Z in $\mathcal{H}_e$ which are $d\pi$ related to X^*, Y^* and Z^*, respectively.

Using (3.12) and Corollary 3.12, we have the following:

Proposition 4.3. *Let G be a Lie group with a bi-invariant metric and G/K be a naturally reductive Riemannian homogeneous space. Then we have*

$$\|A^{\mathcal{V}}(X,Y)\|^2 = \frac{1}{3}\langle[[X,Y],Y],X\rangle \tag{4.22}$$

for any linearly independent vector fields X and Y in $\mathcal{H}_e$.

Proof. Let $X, Y \in \mathcal{H}_e$ and $\Pi = \text{Span}\{X, Y\}$ be a plane section. Under the assumption, since G is a Lie group with a bi-invariant metric, we have

$$K(\Pi) = \frac{1}{4}\langle [[X, Y], Y], X)\rangle. \tag{4.23}$$

Using Proposition 3.2 and Proposition 4.2 in (2.17), we find (4.22). □

Corollary 4.3. *Let π be a mapping from a Lie group with a bi-invariant metric to a naturally reductive Rimannian homogeneous space. Then we have the following statements*

i) G is flat if and only if the tensor $A^{\mathcal{V}}$ vanishes identically.
ii) If π has minimal fibers, then G is flat.

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Affiliations

MEHMET GÜLBAHAR
ADDRESS: Harran University, Department of Mathematics, Şanlıurfa-Turkey
E-MAIL: mehmetgulbahar@harran.edu.tr
ORCID ID: 0000-0001-6950-7633

EROL KILIÇ
ADDRESS: İnönü University, Department of Mathematics, Malatya, Turkey
E-MAIL: erol.kilic@inonu.edu.tr
ORCID ID: 0000-0001-7536-0404

SADIK KELEŞ
ADDRESS: İnönü University, Department of Mathematics, Malatya, Turkey.
E-MAIL: sadik.keles@inonu.edu.tr
ORCID ID: 0000-0003-3981-2092