INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 11 NO. 1 PAGE 93–99 (2018)

On Semi-Tensor Bundle

Furkan Yıldırım*

(Communicated by Arif Salimov)

ABSTRACT

We investigate some lifts of tensor fields of type (1,0) on a cross-section in the semi-tensor (pull-back) bundle tM of tensor bundle TM of type (p,q) by using projection (submersion) of the cotangent bundle T*M and we find some relation for them.

Keywords: *Vector field; complete lift; cross-section; horizontal lift; pull-back bundle; cotangent bundle; semi-tensor bundle*
AMS Subject Classification (2010): *Primary: 53A45; Secondary: 55R10; 57R25.*

1. Introduction

Let M_n be an n-dimensional differentiable manifold of class C^∞, and let $(T^*(M_n), \pi_1, M_n)$ be a cotangent bundle over M_n. We use the notation $(x^i) = \left(x^{\overline{\alpha}}, x^\alpha\right)$, where the indices $i, j, ...$ run from 1 to $2n$, the indices $\overline{\alpha}, \overline{\beta}, ...$ from 1 to n and the indices $\alpha, \beta, ...$ from $n+1$ to $2n$, x^α are coordinates in M_n, $x^{\overline{\alpha}} = p_\alpha$ are fibre coordinates of the cotangent bundle $T^*(M_n)$.

Let now $\left(T^p_q(M_n), \widetilde{\pi}, M_n\right)$ be a tensor bundle [3], [6], [[7], p.118] with base space M_n, and let $T^*(M_n)$ be cotangent bundle determined by a natural projection (submersion) $\pi_1 : T^*(M_n) \to M_n$. The semi-tensor bundle (induced, pull-back [4],[5],[8],[9],[11],[12],[13],[14]) of the tensor bundle $\left(T^p_q(M_n), \widetilde{\pi}, M_n\right)$ is the bundle $\left(t^p_q(M_n), \pi_2, T^*(M_n)\right)$ over cotangent bundle $T^*(M_n)$ with a total space

$$\begin{aligned} t^p_q(M_n) &= \left\{\left(\left(x^{\overline{\alpha}}, x^\alpha\right), x^{\overline{\overline{\alpha}}}\right) \in T^*(M_n) \times \left(T^p_q\right)_x(M_n) : \pi_1\left(x^{\overline{\alpha}}, x^\alpha\right) = \widetilde{\pi}\left(x^\alpha, x^{\overline{\overline{\alpha}}}\right) = (x^\alpha)\right\} \\ &\subset T^*(M_n) \times \left(T^p_q\right)_x(M_n) \end{aligned}$$

and with the projection map $\pi_2 : t^p_q(M_n) \to T^*(M_n)$ defined by $\pi_2(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}}) = \left(x^{\overline{\alpha}}, x^\alpha\right)$, where $\left(T^p_q\right)_x(M_n)\left(x = \pi_1(\widetilde{x}), \widetilde{x} = \left(x^{\overline{\alpha}}, x^\alpha\right) \in T^*(M_n)\right)$ is the tensor space at a point x of M_n, where $x^{\overline{\overline{\alpha}}} = t^{\beta_1 \dots \beta_p}_{\alpha_1 \dots \alpha_q}$ $\left(\overline{\overline{\alpha}}, \overline{\overline{\beta}}, ... = 2n+1, ..., 2n+n^{p+q}\right)$ are fiber coordinates of the tensor bundle $T^p_q(M_n)$.

If $(x^{i'}) = (x^{\overline{\alpha}'}, x^{\alpha'}, x^{\overline{\overline{\alpha}}'})$ is another system of local adapted coordinates in the semi-tensor bundle $t^p_q(M_n)$, then we have

$$\begin{cases} x^{\overline{\alpha}'} = \frac{\partial x^\beta}{\partial x^{\alpha'}} p_\beta, \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right), \\ x^{\overline{\overline{\alpha}}'} = t^{\beta'_1 \dots \beta'_p}_{\alpha'_1 \dots \alpha'_q} = A^{\beta'_1 \dots \beta'_p}_{\alpha_1 \dots \alpha_p} A^{\beta_1 \dots \beta_q}_{\alpha'_1 \dots \alpha'_q} t^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q} = A^{(\beta')}_{(\alpha)} A^{(\beta)}_{(\alpha')} x^{\overline{\beta}}. \end{cases} \tag{1.1}$$

The Jacobian of (1.1) has the components

$$\bar{A} = \left(A^{I'}_J\right) = \begin{pmatrix} A^\beta_{\alpha'} & p_\sigma A^{\beta'}_\beta A^\sigma_{\beta'\alpha'} & 0 \\ 0 & A^{\alpha'}_\beta & 0 \\ 0 & t^{(\alpha)}_{(\sigma)} \partial_\beta A^{(\beta')}_{(\alpha)} A^{(\sigma)}_{(\alpha')} & A^{(\beta')}_{(\alpha)} A^{(\beta)}_{(\alpha')} \end{pmatrix}, \tag{1.2}$$

where $I = (\overline{\alpha}, \alpha, \overline{\overline{\alpha}})$, $J = (\overline{\beta}, \beta, \overline{\overline{\beta}})$, $I, J...=1, ..., 2n+n^{p+q}$, $t^{(\alpha)}_{(\sigma)} = t^{\alpha_1 \dots \alpha_p}_{\sigma_1 \dots \sigma_q}$, $A^{\alpha'}_\beta = \frac{\partial x^{\alpha'}}{\partial x^\beta}$, $A^\beta_{\alpha'} = \frac{\partial x^\beta}{\partial x^{\alpha'}}$, $A^\sigma_{\beta'\alpha'} = \frac{\partial^2 x^\sigma}{\partial x^{\beta'} \partial x^{\alpha'}}$.
It is easily verified that the condition $Det\bar{A} \neq 0$ is equivalent to the condition:

$$Det(A^\beta_{\alpha'}) \neq 0, Det(A^{\alpha'}_\beta) \neq 0, Det(A^{(\beta')}_{(\alpha)} A^{(\beta)}_{(\alpha')}) \neq 0.$$

Received : *17-12-2017,* ***Accepted :*** *20-04-2018*
*** Corresponding author***

Also, $\dim t_q^p(M_n) = 2n + n^{p+q}$.

We note that special class of semi-tensor bundle was examined in [2]. The main purpose of this paper is to study semi-tensor (pull-back) bundle $t_q^p(M_n)$ of tensor bundle $T_q^p(M_n)$ by using projection of the cotangent bundle $T^*(M_n)$.

We denote by $\Im_q^p(T^*(M_n))$ and $\Im_q^p(M_n)$ the modules over $F(T^*(M_n))$ and $F(M_n)$ of all tensor fields of type (p,q) on $T^*(M_n)$ and M_n, respectively, where $F(T^*(M_n))$ and $F(M_n)$ denote the rings of real-valued C^∞ −functions on $T^*(M_n)$ and M_n, respectively.

2. Vertical lifts of tensor fields and $\gamma-$ operator

Let $A \in \Im_q^p(T^*(M_n))$. On putting

$$^{vv}A = \begin{pmatrix} ^{vv}A^{\overline{\alpha}} \\ ^{vv}A^{\alpha} \\ ^{vv}A^{\overline{\overline{\alpha}}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} \end{pmatrix}, \tag{2.1}$$

from (1.2), we easily see that with $^{vv}A' = \bar{A}\,(^{vv}A)$. The vector field $^{vv}A \in \Im_0^1(t_q^p(M_n))$ is called the vertical lift of $A \in \Im_q^p(T^*(M_n))$ to the semi-tensor bundle $t_q^p(M_n)$.

Let $\varphi \in \Im_1^1(M_n)$. We define a vector field $\gamma\varphi$ in $\pi^{-1}(U)$ by

$$\begin{cases} \gamma\varphi = \left(\sum_{\lambda=1}^{p} t^{\alpha_1\ldots\varepsilon\ldots\alpha_p}_{\beta_1\ldots\beta_q}\varphi^{\alpha_\lambda}_{\varepsilon}\right)\frac{\partial}{\partial x^{\overline{\overline{\beta}}}}, & (p\geq 1, q\geq 0) \\ \widetilde{\gamma}\varphi = \left(\sum_{\mu=1}^{q} t^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\varepsilon\ldots\beta_q}\varphi^{\varepsilon}_{\beta_\mu}\right)\frac{\partial}{\partial x^{\overline{\overline{\beta}}}}, & (p\geq 0, q\geq 1) \end{cases} \tag{2.2}$$

with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ on $t_q^p(M_n)$. From (1.2) we easily see that the vector fields $\gamma\varphi$ and $\widetilde{\gamma}\varphi$ defined in each $\pi^{-1}(U) \subset t_q^p(M_n)$ determine respectively global vertical vector fields on $t_q^p(M_n)$. We call $\gamma\varphi$ (or $\widetilde{\gamma}\varphi$) the vertical-vector lift of the tensor field $\varphi \in \Im_1^1(M_n)$ to $t_q^p(M_n)$. For any $\varphi \in \Im_1^1(M_n)$, if we take account of (1.2) and (2.2), we can prove that $(\gamma\varphi)' = \bar{A}\,(\gamma\varphi)$. Where $\gamma\varphi$ is a vector field defined by

$$\gamma\varphi = (\gamma\varphi)^I = \begin{pmatrix} 0 \\ 0 \\ \sum_{\lambda=1}^{p} t^{\alpha_1\ldots\varepsilon\ldots\alpha_p}_{\beta_1\ldots\beta_q}\varphi^{\alpha_\lambda}_{\varepsilon} \end{pmatrix}. \tag{2.3}$$

Let $\varphi \in \Im_1^1(M_n)$. On putting

$$\widetilde{\gamma}\varphi = (\widetilde{\gamma}\varphi)^I = \begin{pmatrix} 0 \\ 0 \\ \sum_{\mu=1}^{q} t^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\varepsilon\ldots\beta_q}\varphi^{\varepsilon}_{\beta_\mu} \end{pmatrix}, \tag{2.4}$$

we easily see that $(\widetilde{\gamma}\varphi)' = \bar{A}\,(\widetilde{\gamma}\varphi)$.

For any $\varphi \in \Im_1^1(T^*(M_n))$, if we take account of (1.2), we can prove that $(\gamma\varphi)' = \bar{A}(\gamma\varphi)$, where $\gamma\varphi$ is a vector field defined by

$$\gamma\varphi = \begin{pmatrix} -p_\sigma F^\sigma_\beta \\ 0 \\ 0 \end{pmatrix}, \tag{2.5}$$

with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$.

3. Complete lifts of vector fields

Let $X \in \Im_0^1(T^*(M_n))$, i.e. $X = X^\alpha\partial_\alpha$. The complete lift ^cX of X to cotangent bundle is defined by $^cX = X^\alpha\partial_\alpha - p_\beta(\partial_\alpha X^\beta)\partial_{\overline{\alpha}}$ [[10], p.236]. On putting

$$^{cc}X = \begin{pmatrix} ^{cc}X^{\overline{\beta}} \\ ^{cc}X^{\beta} \\ ^{cc}X^{\overline{\overline{\beta}}} \end{pmatrix} = \begin{pmatrix} -p_\varepsilon(\partial_\beta X^\varepsilon) \\ X^\beta \\ \sum_{\lambda=1}^{p} t^{\alpha_1\ldots\varepsilon\ldots\alpha_p}_{\beta_1\ldots\beta_q}\partial_\varepsilon X^{\alpha_\lambda} - \sum_{\mu=1}^{q} t^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\varepsilon\ldots\beta_q}\partial_{\beta_\mu}X^\varepsilon \end{pmatrix}, \tag{3.1}$$

from (1.2), we easily see that $^{cc}X' = \bar{A}(^{cc}X)$. The vector field ^{cc}X is called the complete lift of $^cX \in \Im_0^1(T^*(M_n))$ to $t_q^p(M_n)$.

4. Horizontal lifts of vector fields

Let $X \in \Im_0^1(T^*(M_n))$, i.e. $X = X^\alpha(x^\alpha)\partial_\alpha$. If we take account of (1.2), we can prove that ${}^{HH}X' = \bar{A}\left({}^{HH}X\right)$, where ${}^{HH}X \in \Im_0^1(t_q^p(M_n))$ is a vector field defined by

$$ {}^{HH}X = \begin{pmatrix} X^\alpha \Gamma_{\alpha\beta} \\ X^\beta \\ X^l(\sum_{\mu=1}^q \Gamma_{l\beta_\mu}^\varepsilon t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p} - \sum_{\lambda=1}^p \Gamma_{l\varepsilon}^{\alpha_\lambda} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}) \end{pmatrix}, \tag{4.1} $$

with respect to the coordinates $(x^{\overline{\beta}}, x^\beta, x^{\overline{\overline{\beta}}})$ on $t_q^p(M_n)$. We call ${}^{HH}X$ the horizontal lift of the vector field X to $t_q^p(M_n)$, where $\Gamma_{\alpha\beta} = p_\varepsilon \Gamma_{\alpha\beta}^\varepsilon$.

Theorem 4.1. *If $X \in \Im_0^1(T^*(M_n))$ then*

$$ {}^{cc}X - {}^{HH}X = \gamma(\hat{\nabla}X) - \widetilde{\gamma}(\hat{\nabla}X) + \gamma(\nabla X), $$

where the symmetric affine connection $\hat{\nabla}$ is the given by $\widehat{\Gamma}_{\beta\theta}^\alpha = \Gamma_{\theta\beta}^\alpha$.

Proof. From (2.3), (2.4), (2.5), (3.1) and (4.1), we have

$$ \begin{aligned} {}^{cc}X - {}^{HH}X &= \begin{pmatrix} -p_\varepsilon(\partial_\beta X^\varepsilon) \\ X^\beta \\ \sum_{\lambda=1}^p t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p} \partial_\varepsilon X^{\alpha_\lambda} - \sum_{\mu=1}^q t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p} \partial_{\beta_\mu} X^\varepsilon \end{pmatrix} \\ &\quad - \begin{pmatrix} p_\varepsilon X^\alpha \Gamma_{\alpha\beta}^\varepsilon \\ X^\beta \\ X^l(\sum_{\mu=1}^q \Gamma_{l\beta_\mu}^\varepsilon t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p} - \sum_{\lambda=1}^p \Gamma_{l\varepsilon}^{\alpha_\lambda} t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p}) \end{pmatrix} \\ &= \begin{pmatrix} -p_\varepsilon(\partial_\beta X^\varepsilon) - p_\varepsilon X^\alpha \Gamma_{\alpha\beta}^\varepsilon \\ 0 \\ \sum_{\lambda=1}^p t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p} (\partial_\varepsilon X^{\alpha_\lambda} + \Gamma_{l\varepsilon}^{\alpha_\lambda} X^l) - \sum_{\mu=1}^q t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p} (\partial_{\beta_\mu} X^\varepsilon + \Gamma_{l\beta_\mu}^\varepsilon X^l) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \sum_{\lambda=1}^p t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p} \left(\partial_\varepsilon X^{\alpha_\lambda} + \Gamma_{l\varepsilon}^{\alpha_\lambda} X^l\right) \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \sum_{\mu=1}^q t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p} \left(\partial_{\beta_\mu} X^\varepsilon + \Gamma_{l\beta_\mu}^\varepsilon X^l\right) \end{pmatrix} \\ &\quad + \begin{pmatrix} -p_\varepsilon(\partial_\beta X^\varepsilon + X^\alpha \Gamma_{\alpha\beta}^\varepsilon) \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \sum_{\lambda=1}^p t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p} \underbrace{(\partial_\varepsilon X^{\alpha_\lambda} + \widehat{\Gamma}_{\varepsilon l}^{\alpha_\lambda} X^l)}_{\hat{\nabla}_\varepsilon \widetilde{X}^{\alpha_\lambda}} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \sum_{\mu=1}^q t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p} \underbrace{(\partial_{\beta_\mu} X^\varepsilon + \widehat{\Gamma}_{\beta_\mu l}^\varepsilon X^l)}_{\hat{\nabla}_{\beta_\mu} \widetilde{X}^\varepsilon} \end{pmatrix} \\ &\quad + \begin{pmatrix} -p_\varepsilon \underbrace{(\partial_\beta X^\varepsilon + X^\alpha \Gamma_{\alpha\beta}^\varepsilon)}_{\nabla_\beta X^\varepsilon} \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \sum_{\lambda=1}^p t_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\varepsilon\ldots\alpha_p} \left(\hat{\nabla}_\varepsilon \widetilde{X}^{\alpha_\lambda}\right) \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \sum_{\mu=1}^q t_{\beta_1\ldots\varepsilon\ldots\beta_q}^{\alpha_1\ldots\alpha_p} \left(\hat{\nabla}_{\beta_\mu} \widetilde{X}^\varepsilon\right) \end{pmatrix} + \begin{pmatrix} -p_\varepsilon\left(\nabla_\beta X^\varepsilon\right) \\ 0 \\ 0 \end{pmatrix} \\ &= \gamma(\hat{\nabla}_\varepsilon \widetilde{X}^{\alpha_\lambda}) - \widetilde{\gamma}\left(\hat{\nabla}_{\beta_\mu} \widetilde{X}^\varepsilon\right) + \gamma\left(\nabla_\beta X^\varepsilon\right) = \gamma(\hat{\nabla}X) - \widetilde{\gamma}(\hat{\nabla}X) + \gamma(\nabla X), \end{aligned} $$

which prove Theorem 4.1. □

5. Cross-sections in the semi-tensor bundle

Let $\xi \in \Im_q^p(M_n)$ be a tensor field on M_n. Then the correspondence $x \to \xi_x$, ξ_x being the value of ξ at $x \in T^*(M_n)$, determines a cross-section β_ξ of semi-tensor bundle. Thus if $\sigma_\xi : M_n \to T_q^p(M_n)$ is a cross-section of $\left(T_q^p(M_n), \widetilde{\pi}, M_n\right)$, such that $\widetilde{\pi} \circ \sigma_\xi = I_{(M_n)}$, an associated cross-section $\beta_\xi : T^*(M_n) \to t_q^p(M_n)$ of semi-tensor bundle $\left(t_q^p(M_n), \pi_2, T^*(M_n)\right)$ defined by [[1], p. 217-218], [4], [5], [[10].p. 301]:

$$\beta_\xi\left(x^{\overline{\alpha}}, x^\alpha\right) = \left(x^{\overline{\alpha}}, x^\alpha, \sigma_\xi \circ \pi_1\left(x^{\overline{\alpha}}, x^\alpha\right)\right) = \left(x^{\overline{\alpha}}, x^\alpha, \sigma_\xi\left(x^\alpha\right)\right) = \left(x^{\overline{\alpha}}, x^\alpha, \xi_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\left(x^\beta\right)\right).$$

If the tensor field ξ has the local components $\xi_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\left(x^\beta\right)$, the cross-section $\beta_\xi\left(T^*(M_n)\right)$ of $t_q^p(M_n)$ is locally expressed by

$$\begin{cases} x^{\overline{\beta}} = p_\beta = \theta_\beta\left(x^\alpha\right), \\ x^\beta = x^\beta, \\ x^{\overline{\overline{\beta}}} = \xi_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\alpha_p}\left(x^\alpha\right), \end{cases} \tag{5.1}$$

with respect to the coordinates $x^B = (x^{\overline{\beta}}, x^\beta, x^{\overline{\overline{\beta}}})$ in $t_q^p(M_n)$.
$x^{\overline{\alpha}} = p_\alpha$ being considered as parameters. Thus, by differentiating with respect to p_α, we easily see that the n local vector fields $B_{(\overline{\theta})}$ $(\overline{\theta} = 1, ..., n)$ with components

$$B_{(\overline{\theta})} : \left(B_{(\overline{\theta})}^B\right) = \partial_{(\overline{\theta})} x^B = \begin{pmatrix} \partial_{\overline{\theta}}\theta_\beta \\ \partial_{\overline{\theta}} x^\beta \\ \partial_{\overline{\theta}}\xi_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\alpha_p} \end{pmatrix} = \begin{pmatrix} \delta_\beta^\theta \\ 0 \\ 0 \end{pmatrix}$$

is tangent to the fibre, where

$$\delta_\beta^\theta = A_\beta^\theta = \frac{\partial x^\theta}{\partial x^\beta}.$$

Let ω be an 1-form with local components ω_β on M_n, so that ω is a 1-form with local expression $\omega = \omega_\beta dx^\beta$. We denote by $B\omega$ the vector field with local components

$$B\omega : \left(B_{(\overline{\theta})}^B \omega_\theta\right) = \begin{pmatrix} \omega_\beta \\ 0 \\ 0 \end{pmatrix}, \tag{5.2}$$

which is tangent to the fibre.

Taking the derivative with respect to x^θ, we have vector fields $C_{(\theta)}$ $(\theta = n+1, ..., 2n)$ with components

$$C_{(\theta)} = \frac{\partial x^B}{\partial x^\theta} = \partial_\theta x^B = \begin{pmatrix} \partial_\theta\theta_\beta \\ \partial_\theta x^\beta \\ \partial_\theta\xi_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\alpha_p} \end{pmatrix},$$

which are tangent to the cross-section $\beta_\xi\left(T^*(M_n)\right)$.

Thus $C_{(\theta)}$ has the components

$$C_{(\theta)} : \left(C_{(\theta)}^B\right) = \begin{pmatrix} \partial_\theta\theta_\beta \\ \delta_\theta^\beta \\ \partial_\theta\xi_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\alpha_p} \end{pmatrix},$$

with respect to the coordinates $(x^{\overline{\beta}}, x^\beta, x^{\overline{\overline{\beta}}})$ in $t_q^p(M_n)$. Where

$$\delta_\theta^\beta = A_\theta^\beta = \frac{\partial x^\beta}{\partial x^\theta}.$$

Let $X \in \Im_0^1\left(T^*(M_n)\right)$. Then we denote by CX the vector field with local components

$$CX : \left(C_{(\theta)}^B X^\theta\right) = \begin{pmatrix} X^\theta\partial_\theta\theta_\beta \\ X^\beta \\ X^\theta\partial_\theta\xi_{\beta_1\ldots\beta_q}^{\alpha_1\ldots\alpha_p} \end{pmatrix}, \tag{5.3}$$

with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ in $t^p_q(M_n)$, which is defined globally along $\beta_\xi\,(T^*(M_n))$.

On the other hand, the fibre is locally expressed by

$$\begin{cases} x^{\overline{\beta}} = p_\beta = const., \\ x^{\beta} = const., \\ x^{\overline{\overline{\beta}}} = t^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} = t^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q}, \end{cases}$$

$t^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q}$ being considered as parameters. Thus, by differentiating with respect to $x^{\overline{\overline{\beta}}} = t^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q}$, we easily see that the vector fields $E_{\left(\overline{\overline{\theta}}\right)}$ $(\overline{\theta} = 2n+1, ..., 2n+n^{p+q})$ with components

$$E_{\left(\overline{\overline{\theta}}\right)} : \left(E^B_{\left(\overline{\overline{\theta}}\right)}\right) = \partial_{\overline{\overline{\theta}}} x^B = \begin{pmatrix} \partial_{\overline{\overline{\theta}}} \theta_\beta \\ \partial_{\overline{\overline{\theta}}} x^\beta \\ \partial_{\overline{\overline{\theta}}} t^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \delta^{\theta_1}_{\beta_1} \ldots \delta^{\theta_q}_{\beta_q} \delta^{\alpha_1}_{\gamma_1} \ldots \delta^{\alpha_p}_{\gamma_p} \end{pmatrix}$$

is tangent to the fibre, where δ is the Kronecker symbol.

Let ξ be a tensor field of type (p,q) with local components

$$\xi = \xi^{\gamma_1 \ldots \gamma_p}_{\theta_1 \ldots \theta_q} dx^{\theta_1} \otimes \ldots \otimes dx^{\theta_q} \otimes \partial_{\gamma_1} \otimes \ldots \otimes \partial_{\gamma_p}$$

on M_n.

We denote by $E\xi$ the vector field with local components

$$E\xi : \left(E^B_{\left(\overline{\overline{\theta}}\right)} \xi^{\gamma_1 \ldots \gamma_p}_{\theta_1 \ldots \theta_q}\right) = \begin{pmatrix} 0 \\ 0 \\ \xi^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} \end{pmatrix}, \tag{5.4}$$

which is tangent to the fibre.

Theorem 5.1. *Let* $\psi,\omega \in \Im^0_1(M_n)$ *. For the Lie product, we have*

$$[B\psi, B\omega] = 0.$$

Proof. If $\psi, \omega \in \Im^0_1(M_n)$ and $\begin{pmatrix} [B\psi, B\omega]^{\overline{\beta}} \\ [B\psi, B\omega]^{\beta} \\ [B\psi, B\omega]^{\overline{\overline{\beta}}} \end{pmatrix}$ are the components of $[B\psi, B\omega]$ with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ in $t^p_q(M_n)$, then we have

$$\begin{aligned} [B\psi, B\omega]^J &= \psi^I \partial_I \omega^J - \omega^I \partial_I \psi^J \\ &= \psi^{\overline{\alpha}} \partial_{\overline{\alpha}} \omega^J + \psi^{\alpha} \partial_{\alpha} \omega^J + \psi^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} \omega^J - \omega^{\overline{\alpha}} \partial_{\overline{\alpha}} \psi^J - \omega^{\alpha} \partial_{\alpha} \psi^J - \omega^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} \psi^J \\ &= \psi_\alpha \partial_{\overline{\alpha}} \omega^J - \omega_\alpha \partial_{\overline{\alpha}} \psi^J. \end{aligned}$$

Firstly, if $J = \overline{\beta}$, we have

$$\begin{aligned} [B\psi, B\omega]^{\overline{\beta}} &= \psi_\alpha \partial_{\overline{\alpha}} \omega^{\overline{\beta}} - \omega_\alpha \partial_{\overline{\alpha}} \psi^{\overline{\beta}} \\ &= \psi_\alpha \partial_{\overline{\alpha}} \omega_\beta - \omega_\alpha \partial_{\overline{\alpha}} \psi_\beta \\ &= 0 \end{aligned}$$

by virtue of (5.2). Secondly, if $J = \beta$, we have

$$\begin{aligned} [B\psi, B\omega]^{\beta} &= \psi_\alpha \partial_{\overline{\alpha}} \omega^{\beta} - \omega_\alpha \partial_{\overline{\alpha}} \psi^{\beta} \\ &= 0 \end{aligned}$$

by virtue of (5.2). Thirdly, if $J = \overline{\overline{\beta}}$. Then we have

$$\begin{aligned} [B\psi, B\omega]^{\overline{\overline{\beta}}} &= \psi_\alpha \partial_{\overline{\alpha}} \omega^{\overline{\overline{\beta}}} - \omega_\alpha \partial_{\overline{\alpha}} \psi^{\overline{\overline{\beta}}} \\ &= 0 \end{aligned}$$

by virtue of (5.2). Thus, we have $[B\psi, B\omega] = 0$. □

Theorem 5.2. *Let X be a vector field on $T^*(M_n)$, we have along $\beta_\xi\left(T^*(M_n)\right)$ the formula*

$$^{cc}X = -B\left(L_X\theta\right) + CX + E\left(-L_X\xi\right),$$

where $L_X\theta$ denotes the Lie derivative of θ with respect to X, and $L_X\xi$ denotes the Lie derivative of ξ with respect to X.

Proof. Using (3.1), (5.2), (5.3) and (5.4), we have

$$\begin{aligned}
-B\left(L_X\theta\right) + CX + E\left(-L_X\xi\right) &= -\begin{pmatrix} X^\theta\partial_\theta\theta_\beta + \theta_\theta\partial_\beta X^\theta \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} X^\theta\partial_\theta\theta_\beta \\ X^\beta \\ X^\theta\partial_\theta\xi^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} \end{pmatrix} \\
&+ \begin{pmatrix} 0 \\ 0 \\ -X^\theta\partial_\theta\xi^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} - \sum_{\mu=1}^{q}\partial_{\beta_\mu}X^\beta\xi^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta\ldots\beta_q} + \sum_{\lambda=1}^{p}\partial_\beta X^{\alpha_\lambda}\xi^{\alpha_1\ldots\varepsilon\ldots\alpha_p}_{\beta_1\ldots\beta_q} \end{pmatrix} \\
&= \begin{pmatrix} -\theta_\theta\left(\partial_\beta X^\theta\right) \\ X^\beta \\ -\sum_{\mu=1}^{q}\partial_{\beta_\mu}X^\beta\xi^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta\ldots\beta_q} + \sum_{\lambda=1}^{p}\partial_\beta X^{\alpha_\lambda}\xi^{\alpha_1\ldots\varepsilon\ldots\alpha_p}_{\beta_1\ldots\beta_q} \end{pmatrix} \\
&= {}^{cc}X.
\end{aligned}$$

Thus, we have Theorem 5.2. □

On the other hand, on putting $C_{\left(\overline{\overline{\beta}}\right)} = E_{\left(\overline{\overline{\beta}}\right)}$, we write the adapted frame of $\beta_\xi\left(T^*(M_n)\right)$ as $\left\{B_{\left(\overline{\beta}\right)}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)}\right\}$. The adapted frame $\left\{B_{\left(\overline{\beta}\right)}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)}\right\}$ of $\beta_\xi\left(T^*(M_n)\right)$ is given by the matrix

$$\widetilde{A} = \left(\widetilde{A}^A_B\right) = \begin{pmatrix} \delta^\beta_\alpha & \partial_\beta\theta_\alpha & 0 \\ 0 & \delta^\alpha_\beta & 0 \\ 0 & \partial_\beta\xi^{\sigma_1\ldots\sigma_p}_{\alpha_1\ldots\alpha_q} & \delta^{\beta_1}_{\alpha_1}\ldots\delta^{\beta_q}_{\alpha_q}\delta^{\sigma_1}_{\gamma_1}\ldots\delta^{\sigma_p}_{\gamma_p} \end{pmatrix}. \tag{5.5}$$

Since the matrix $\widetilde{A}$ in (5.5) is non-singular, it has the inverse. Denoting this inverse by $\left(\widetilde{A}\right)^{-1}$, we have

$$\left(\widetilde{A}\right)^{-1} = \left(\widetilde{A}^B_C\right)^{-1} = \begin{pmatrix} \delta^\theta_\beta & -\partial_\theta\theta_\beta & 0 \\ 0 & \delta^\beta_\theta & 0 \\ 0 & -\partial_\theta\xi^{\sigma_1\ldots\sigma_p}_{\beta_1\ldots\beta_q} & \delta^{\theta_1}_{\beta_1}\ldots\delta^{\theta_q}_{\beta_q}\delta^{\sigma_1}_{\gamma_1}\ldots\delta^{\sigma_p}_{\gamma_q} \end{pmatrix}, \tag{5.6}$$

where $\widetilde{A}\left(\widetilde{A}\right)^{-1} = (\widetilde{A}^A_B)\left(\widetilde{A}^B_C\right)^{-1} = \delta^A_C = \widetilde{I}$, where $A = (\overline{\alpha}, \alpha, \overline{\overline{\alpha}})$, $B = \left(\overline{\beta}, \beta, \overline{\overline{\beta}}\right)$, $C = \left(\overline{\theta}, \theta, \overline{\overline{\theta}}\right)$.

Proof. In fact, from (5.5) and (5.6), we easily see that

$$\begin{aligned}
\widetilde{A}\left(\widetilde{A}\right)^{-1} &= (\widetilde{A}^A_B)\left(\widetilde{A}^B_C\right)^{-1} \\
&= \begin{pmatrix} \delta^\beta_\alpha & \partial_\beta\theta_\alpha & 0 \\ 0 & \delta^\alpha_\beta & 0 \\ 0 & \partial_\beta\xi^{\sigma_1\ldots\sigma_p}_{\alpha_1\ldots\alpha_q} & \delta^{\beta_1}_{\alpha_1}\ldots\delta^{\beta_q}_{\alpha_q}\delta^{\sigma_1}_{\gamma_1}\ldots\delta^{\sigma_p}_{\gamma_p} \end{pmatrix}\begin{pmatrix} \delta^\theta_\beta & -\partial_\theta\theta_\beta & 0 \\ 0 & \delta^\beta_\theta & 0 \\ 0 & -\partial_\theta\xi^{\sigma_1\ldots\sigma_p}_{\beta_1\ldots\beta_q} & \delta^{\theta_1}_{\beta_1}\ldots\delta^{\theta_q}_{\beta_q}\delta^{\sigma_1}_{\gamma_1}\ldots\delta^{\sigma_p}_{\gamma_q} \end{pmatrix} \\
&= \begin{pmatrix} \delta^\theta_\alpha & \partial_\theta\theta_\alpha - \partial_\theta\theta_\alpha & 0 \\ 0 & \delta^\alpha_\theta & 0 \\ 0 & \partial_\theta\xi^{\sigma_1\ldots\sigma_p}_{\alpha_1\ldots\alpha_q} - \partial_\theta\xi^{\sigma_1\ldots\sigma_p}_{\alpha_1\ldots\alpha_q} & \delta^{\theta_1}_{\alpha_1}\ldots\delta^{\theta_q}_{\alpha_q} \end{pmatrix} = \begin{pmatrix} \delta^\theta_\alpha & 0 & 0 \\ 0 & \delta^\alpha_\theta & 0 \\ 0 & 0 & \delta^\theta_\alpha \end{pmatrix} = \delta^A_C = \widetilde{I}.
\end{aligned}$$

□

Then we see from Theorem 5.2 that the complete lift ^{cc}X of a vector field $X \in \Im^1_0(T^*(M_n))$ has along $\beta_\xi\left(T^*(M_n)\right)$ components of the form

$$^{cc}X : \begin{pmatrix} -L_X\theta \\ X \\ -L_X\xi \end{pmatrix},$$

with respect to the adapted frame $\left\{B_{(\overline{\beta})}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)}\right\}$.

Let $A \in \Im_q^p(T^*(M_n))$. If we take account of (2.1) and (5.5), we can easily prove that ${}^{vv}A' = \widetilde{A}\left({}^{vv}A\right)$, where ${}^{vv}A \in \Im_0^1(t_q^p(M_n))$ is a vector field defined by

$$ {}^{vv}A = \begin{pmatrix} {}^{vv}A^{\overline{\alpha}} \\ {}^{vv}A^{\alpha} \\ {}^{vv}A^{\overline{\overline{\alpha}}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ A^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} \end{pmatrix}, $$

with respect to the adapted frame $\left\{B_{(\overline{\beta})}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)}\right\}$ of $\beta_\xi\left(T^*(M_n)\right)$.

$B\omega$, CX and $E\xi$ also have the components:

$$ B\omega = \begin{pmatrix} \omega_\alpha \\ 0 \\ 0 \end{pmatrix}, CX = \begin{pmatrix} 0 \\ X^\alpha \\ 0 \end{pmatrix}, E\xi = \begin{pmatrix} 0 \\ 0 \\ \xi^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} \end{pmatrix} $$

respectively, with respect to the adapted frame $\left\{B_{(\overline{\beta})}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)}\right\}$ of the cross-section $\beta_\xi\left(T^*(M_n)\right)$ determined by a tensor field ξ of type (p,q) in $T^*(M_n)$.

References

[1] Isham, C.J., Modern Differential Geometry for Physicists. World Scientific, 1999.
[2] Fattaev, H., The lifts of vector fields to the semitensor bundle of the Type (2, 0). *Journal of Qafqaz University*, **25** (2009), no. 1, 136-140.
[3] Gezer, A., Salimov, A.A., Almost complex structures on the tensor bundles. *Arab. J. Sci. Eng. Sect. A Sci.*, **33** (2008), no. 2, 283-296.
[4] Husemoller, D., Fibre Bundles. Springer, New York, 1994.
[5] Lawson, H.B. and Michelsohn, M.L., Spin Geometry. Princeton University Press., Princeton, 1989.
[6] Ledger, A.J. and Yano, K., Almost complex structure on tensor bundles. *J. Dif. Geom.*, **1** (1967), 355-368.
[7] Salimov, A., Tensor Operators and their Applications. Nova Science Publ., New York, 2013.
[8] Salimov, A.A. and Kadıoğlu, E., Lifts of derivations to the semitangent bundle. *Turk J. Math.*, **24** (2000), 259-266.
[9] Steenrod, N., The Topology of Fibre Bundles. Princeton University Press., Princeton, 1951.
[10] Yano, K. and Ishihara, S., Tangent and Cotangent Bundles. Marcel Dekker, Inc., New York, 1973.
[11] Yıldırım, F., On a special class of semi-cotangent bundle, Proceedings of the Institute of Mathematics and Mechanics, (ANAS) **41** (2015), no. 1, 25-38.
[12] Yıldırım, F. and Salimov A., Semi-cotangent bundle and problems of lifts. *Turk J. Math.*, **38** (2014), 325-339.
[13] Yıldırım, F., Note on the cross-section in the semi-tensor bundle. *New Trends in Mathematical Sciences*, **5** (2017) , no. 2, 212-221.
[14] Yıldırım, F., Asl, M.B. and Jabrailzade, F., Vector and affinor fields on cross-sections in the semi-cotangent bundle. *Proceedings of the Institute of Mathematics and Mechanics*, (ANAS) **43** (2017), no. 2, 305-315.

Affiliations

FURKAN YILDIRIM
ADDRESS: Atatürk University, Department of Mathematics, Faculty of Sci. Narman Vocational Training School, 25530, Erzurum Turkey.
E-MAIL: furkan.yildirim@atauni.edu.tr
ORCID ID : orcid.org/0000-0003-0081-7857