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# A Geometric Interpretation of Cauchy-Schwarz Inequality in Terms of Casorati Curvature

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### ABSTRACT

**In a visionary short paper published in 1855, Ossian Bonnet derived a theorem relating prescribed curvature conditions to the admissible maximal length of geodesics on a surface. Bonnet's work opened the pathway for the quest of further connections between curvature conditions and other geometric properties of surfaces, hypersurfaces or Riemannian manifolds. The classical Myers' Theorem in Riemannian geometry provides sufficient conditions for the compactness of a Riemannian manifold in terms of Ricci curvature. In the present work, we are proving a theorem involving sufficient conditions for a smooth hypersurface in Euclidean ambient space to be convex, and the argument relies on an application of Cauchy-Schwarz inequality. This statement represents, in consequence, a geometric interpretation of Cauchy-Schwarz inequality. The curvature conditions are prescribed in terms of Casorati curvature.**

***Keywords:*** *principal curvatures; Casorati curvature; smooth surfaces; smooth hypersurfaces; Cauchy-Schwarz inequality.*
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It might be surprising that a fundamental application of Cauchy-Schwarz inequality yields a remarkable property for smooth hypersurfaces, in which a quantitative piece of information determines a qualitative assertion. We start our investigation by describing first the algebraic part of the result.

We start our note by stating the following.

**Claim 1.** *If for any real numbers $a_1, a_2, a_3$ we have*

$$\max\left\{\sqrt{2(a_1^2+a_2^2)}, \sqrt{2(a_2^2+a_3^2)}, \sqrt{2(a_3^2+a_1^2)}\right\} \leq a_1+a_2+a_3,$$

*then $a_1 \geq 0, a_2 \geq 0,$ and $a_3 \geq 0$.*

To prove this Claim, solve the inequality $\sqrt{2(a_1^2+a_2^2)} \leq a_1+a_2+a_3$, for $a_3$. Then we convert $a_1$ and $a_2$ into polar coordinates. At $r=0$ the property holds trivially. For $r \neq 0$, we have $a_1 = r\cos\theta$, $a_2 = r\sin\theta$. This inequality becomes

$$a_3 = \sqrt{2(a_1^2+a_2^2)} - (a_1+a_2) = r(\sqrt{2} - cos\theta - \sin\theta) \geq 0.$$

The equality holds at $\theta = \frac{\pi}{4}$ or $\theta = \frac{5\pi}{4}$. By repeating the argument twice, we see that equality holds only for $a_1 = a_2 = a_3$. □

This assertion can be generalized as follows.

**Claim 2.** *If for $n \geq 3$ real numbers $a_1, a_2, ..., a_n$ the following inequalities hold:*

$$\sqrt{(n-1)(a_1^2+a_2^2+...+a_{n-1}^2)} \leq a_1+a_2+...+a_n,$$

$$\sqrt{(n-1)(a_1^2+a_2^2+...+a_{n-2}^2+a_n^2)} \leq a_1+a_2+...+a_n,$$

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$$\dots$$
$$\sqrt{(n-1)(a_2^2+a_3^2+...a_n^2)} \leq a_1+a_2+...+a_n.$$

*Then we must have* $a_1 \geq 0, a_2 \geq 0, ... \; a_n \geq 0.$

For a proof, we solve in the first inequality for $a_n$, which appears only in the right hand side term:

$$a_n \geq \sqrt{(n-1)(a_1^2+a_2^2+...+a_{n-1}^2)} - (a_1+a_2+...+a_{n-1}).$$

We need to prove this term is greater than or at least equal to zero. Then we have:

$$a_n \geq \sqrt{(n-1)(a_1^2+a_2^2+...+a_{n-1}^2)} - (a_1+a_2+...+a_{n-1}).$$

We need to prove the right side of this inequality is greater than or at least equal to zero. This is just the Cauchy-Schwarz inequality applied to the numbers $a_1, a_2, ..., a_{n-1}$, and then to $(n-1)$ copies of $1:$

$$\sqrt{(1^2+1^2+...+1^2)(a_1^2+a_2^2+...+a_{n-1}^2)} \geq (1\cdot a_1 + 1\cdot a_2 + ... + 1\cdot a_{n-1}).$$

This proves that $a_n \geq 0$. Similarly one may prove all the other inequalities $a_i \geq 0, i = 1, ..., n-1$. Equality holds when all the $n$ numbers $a_i$, $i = 1, ..., n$ take the same value. □

This seems to be an elementary property of real numbers. However, there is more to this claim than it seems at the first sight.

Pursuing the extension of an interesting theorem for surfaces originally stated and proved by O. Bonnet [1], the classic S. B. Myers' theorem [9] asserts that a complete Riemannian manifold $M$ that satisfies at every point $p \in M$ the condition $Ric_p(v,v) \geq (n-1)r^{-2} > 0$, for any unit vector $v \in T_pM$, is compact and its diameter must be less than or at most equal to $\pi r$. The condition $Ric_p(v,v) \geq 0$ everywhere and a Ricci curvature condition along geodesic rays from a point $p_0 \in M$ has been studied by Calabi in [2]. An interesting classical application to relativity appears in [8]. For some other references on the topic one may see e.g. [7].

This long history of the investigations of all connections between curvature and topology inspire the following

**Question 1.** *Given certain curvature restrictions, do they have any geometric consequences for a given class of geometric objects?*

Additionally, Casorati introduced in 1890 what is today called the Casorati curvature [3]. Let $M^n$ be a smooth hypersurface in the Euclidean ambient space $\mathbb{R}^{n+1}$. Our aim is to show that actually the Claim investigated above and which reduces to a straightforward application of Cauchy-Schwarz inequality, represents a Bonnet-Myers type theorem with Casorati curvatures, in the sense that prescribed curvature conditions imply a geometric property of a hypersurface.

To recall a few concepts in the geometry of differential hypersurfaces or smooth hypersurfaces, let $\sigma : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a smooth hypersurface given by the smooth map $\sigma$. Let $p$ be a point on the hypersurface. Denote $\sigma_k(p) = \frac{\partial \sigma}{\partial x_k}$, for all $k$ from 1 to $n$. Consider $\{\sigma_1(p), \sigma_2(p), ..., \sigma_n(p), N(p)\}$, the Gauss frame of the hypersurface, where $N$ denotes the normal vector field. We denote by $g_{ij}(p)$ the coefficients of the first fundamental form and by $h_{ij}(p)$ the coefficients of the second fundamental form. Then

$$g_{ij}(p) = \langle \sigma_i(p), \sigma_j(p) \rangle, \qquad h_{ij}(p) = \langle N(p), \sigma_{ij}(p) \rangle.$$

The Weingarten map $L_p = -dN_p \circ d\sigma_p^{-1} : T_{\sigma(p)}\sigma \to T_{\sigma(p)}\sigma$ is linear. Denote by $(h_j^i(p))_{1\leq i,j\leq n}$ the matrix associated to Weingarten's map, that is:

$$L_p(\sigma_i(p) = h_i^k(p)\sigma_k(p),$$

where the repeated index and upper script above indicates Einstein's summation convention. Weingerten's operator is self-adjoint, which implies that the roots of the algebraic equation

$$\det(h_j^i(p) - \lambda(p)\delta_j^i) = 0$$

are real. The eigenvalues of Weingarten's linear map are called principal curvatures of the hypersurface. They are the roots $k_1(p), k_2(p), ..., k_n(p)$ of this algebraic equation. The mean curvature at the point $p$ is

$$H(p) = \frac{1}{n}[k_1(p) + ... + k_n(p)],$$

and the Gauss-Kronecker curvature is

$$K(p) = k_1(p)k_2(p)...k_n(p).$$

If all the principal curvature of a smooth regular hypersurface are $\geq 0$, then the hypersurface is convex.

In order to pursue our investigation, we let the principal curvatures play the part of the variables $a_1, ..., a_n$ in Claim 2 above.

One may define the Casorati curvature in the direction of the $k-$dimensional planar section $V_p \subset T_pM$, or the Casorati curvature of order $k$, by

$$C_k(e_1, e_2, ..., e_k) = a_1^2 + ... + a_k^2,$$

for $k \leq n$, where $a_1, ..., a_k$ are the principal curvatures at point $p \in M$. Casorati curvatures are related to the geometry of submanifolds [4, 5, 6], and any information that relates curvature invariants to the topology of the submanifolds is of interest.

In this context, we state the following.

**Question 2.** *Are there any prescribed conditions in terms of Casorati curvature that yield global conclusions about the geometry of a surface or of a hypersurface?*

The following two propositions are inspired by Questions 1 and 2.

**Proposition 0.1.** *Let $\sigma : U \subset \mathbb{R}^3 \to \mathbb{R}^4$ be a regular smooth hypersurface, $Im\ \sigma = M^3$. Let $a_1, a_2, a_3$ be the principal curvatures at $p \in M$. If all the Casorati curvatures of order 2 satisfy with respect to the mean curvature $H(p) = \frac{1}{3}(a_1 + a_2 + a_3)$ the inequality*

$$\sqrt{2C_2(p)} \leq 3H(p),$$

*for every $p$ in $M$, then the hypersurface must be convex.*

*Proof:* The proof is a straightforward application of Claim 1. □

**Proposition 0.2.** *Let $\sigma : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a regular smooth hypersurface, $Im\ \sigma = M^n$. Let $a_1, a_2, ..., a_n$ be the principal curvatures at $p \in M$. If all the Casorati curvatures of order 2 satisfy the inequality*

$$\sqrt{(n-1)C_{n-1}(p)} \leq nH(p),$$

*for every $p$ in $M$, then the hypersurface must be convex.*

*Proof:* The proof is directly Claim 2. □

The above two Propositions provide a geometric interpretation of the algebraic statements investigated in the first two Claims in this paper.

Note that the classical Myers' Theorem [9] asserts the sufficient conditions to determine the compactness of the Riemannian manifold. In the geometric interpretation developed in our note, the statement yields just the convexity of the hypersurface.

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