

Almost Ricci Solitons and Physical Applications

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ABSTRACT

In this paper, we establish a link between a "curvature inheritance symmetry" of a semi-Riemannian manifold and a class of almost Ricci solitons (ARS). In support of this link we present three mathematical models of conformally flat ARS-manifolds. As an application to relativity, by investigating the kinematic and dynamic properties of ARS-spacetimes we present a physical model of three classes (namely, shrinking, steady and expanding) of perfect fluid solutions for ARS-spacetimes and prove the existence of a family of totally umbilical ARS Einstein hypersurfaces of a GRW-spacetime. Finally, we propose two open problems for further study.

Keywords: Ricci solitons, curvature symmetry, totally umbilical, Einstein spaces.

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1. Introduction

In 1982 Hamilton [15] introduced the concept of Ricci flow geometric evolution equation in which one starts with a smooth n -dimensional Riemannian manifold $(M, g(0))$ that evolves its metric by the following equation

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} \quad (i, j = 1, \dots, n), \quad (1.1)$$

where R_{ij} is the Ricci tensor of the metric g_{ij} . Suppose $(M, g(t))$ is a solution of the Ricci flow equation (1.1) on a time interval (α, ω) including $g(0)$. Then, we say that $g(t)$ is a "Ricci soliton" if there exist scalars $\phi(t)$ and a 1-parameter family of diffeomorphisms $\psi(t) : M \rightarrow M$ such that

$$g(t) = \phi(t)\psi(t)^*g(0), \quad (1.2)$$

where $\phi'(t) = -2\lambda$ for a constant λ for all $t \in (\alpha, \omega)$. Differentiating (1.2) with respect to t and using (1.1) we get the following Ricci soliton evolution equation

$$\mathcal{L}_V g_{ij} = 2\lambda g_{ij} - 2R_{ij}. \quad (1.3)$$

where V is a vector field of M . For details on above equation, see [6, Lemma 2.4, page 23]. A solution of (1.3) addresses the following fundamental question:

If $g(0)$ is a complete locally homogeneous metric, how will $g(t)$ evolve?

To deal with above question, Hamilton introduced three special classes of Ricci soliton solutions, namely, shrinking ($\lambda > 0$) that which exists on a maximal time interval $-\infty < t < \omega$ where $\omega < \infty$, steady ($\lambda = 0$) that which exists for all time or expanding ($\lambda < 0$) that which exists on maximal time interval $\alpha < t < \infty$ where $\alpha > -\infty$. These classes yield examples of *ancient, eternal and immortal* solutions, respectively. Warped products provide examples of solitons, such as the well-known eternal solution of the cigar metric which is defined on the plane by $g(t) = \frac{dx^2 + dy^2}{e^{4t + x^2 + y^2}}$ whose scalar curvature $r = \frac{4}{1 + x^2 + y^2} = 4\operatorname{sech}^2 s$, where s is the distance to the origin. This metric is conformal to the Euclidean metric, positively curved, asymptotic to a cylinder and r

decays exponentially fast. Physicists know this example as Witten’s black hole cited in [26]. Actually, in the physics literature, Ricci solitons were first introduced as quasi-Einstein metrics. The shrinking round sphere is the ancient solution of the Ricci flow. Also, for $t > 0$, an expanding immortal solution is a smooth complete metric on R^2 with positive curvature that decays exponentially with respect to the distance from the origin. These and other special examples of above three classes are discussed in [6, Chapter 2]. If V is the gradient of a smooth function f , one can replace V by ∇f and then (M, g, λ, V) is called a gradient Ricci soliton manifold for which the evolution equation (1.3) assumes the form.

$$\nabla_i \nabla_j f + R_{ij} = \lambda g_{ij}. \tag{1.4}$$

Again, cigar solution is the unique rotationally symmetric gradient Ricci soliton eternal solution. Hamilton’s work has been used in resolving many longstanding open problems in Riemannian geometry and 3-dimensional topology. Basic details on this area of research are available in Chow-Knopf [6] and Cao et al. [7] and more cited therein. There are few papers on Ricci solitons for semi-Riemannian (in particular, Lorentzian) manifolds (for example, see Crasmareanu [8], Brozos-Vázquez et al.[5], Duggal [11] and Onda [21].

Recently, Pigola et al.[23] introduced a modified class of the Ricci soliton(RS) evolution equation (1.3) by replacing the soliton constant λ with a variable function Φ and then (M, g, Φ, V) is called an “almost Ricci soliton” manifold, which we denote by ARS-manifold and V the “almost Ricci soliton vector”, briefly denoted by ARS vector such that the evolution equation (1.3) becomes

$$\mathcal{L}_V g_{ij} = 2\Phi g_{ij} - 2R_{ij}. \tag{1.5}$$

For the ARS solution of Ricci flow we consider

$$g(t) = \phi(t, x^a) \psi(t)^* g(0), \tag{1.6}$$

where $\psi(t)$ are diffeomorphisms of M generated by a family of vector fields $X(t)$ and $\phi(t, x^a)$ are point wise scaling functions depending on all the coordinates (t, x^a) of points with the initial condition: $g_{ij}(0) = g_{ij}$, $\psi(0) = I \rightarrow \phi(t, x^a) = 1$. Differentiating (1.6) with respect to t and using the Ricci flow equation (1.1) we get

$$\left(\frac{\partial}{\partial t} \phi(t, x^a) \right)_{|t=0} g_{ij} + \mathcal{L}_{X(0)} g_{ij} = -2R_{ij}. \tag{1.7}$$

Labelling $X(0) = V$ and $\left(\frac{\partial}{\partial t} \phi(t, x^a) \right)_{|t=0} = -2\Phi$ we get the almost Ricci soliton evolution equation (1.5). Just like the case of Ricci solitons, the almost Ricci soliton solution is shrinking (ancient), steady (eternal) or expanding (immortal) according as Φ is positive, zero, or negative, respectively. We say that an ARS-manifold is shrinking (ancient), steady (eternal) or expanding (immortal), in the same order. For an example of an ARS-manifold we refer Barros-Riberiro [4]. If V is the gradient of a smooth function f , we replace V by ∇f . Then, (M, g, Φ, V) is called a gradient ARS-manifold for which the evolution equation (1.4) holds if we replace λ by Φ . Barros et al.[2] have proved that if $n > 2$ and the scalar curvature is constant, then (M, g) is isomorphic to a Euclidean sphere and ARS-manifold is gradient. For an ARS-manifold (M, g, Φ, V) , with $\dim(M) > 2$, and V homothetic, g is Einstein and therefore $\Phi = \lambda$ so ARS-manifold is Ricci soliton(RS). Also, for an ARS-manifold the vector V is conformal if and only if g is Einstein. So far we know following references on ARS-manifolds: Pigola et al.[23], Barros et al.[2], Barros-Riberiro [4], Sharma [24], Wang [25], Duggal [11] and some more referred therein.

In this paper, we let (M, g, V, Φ) be an almost Ricci soliton(ARS) semi-Riemannian manifold for which the evolution equation (1.5) holds. Modified theory of ARS-manifolds has just started which has wide open questions of finding examples of ARS-solutions and an analysis on the existence of their three classes (shrinking, steady, and expanding) for Riemannian and semi-Riemannian manifolds. Motivated by this scope of research, in Section 2 we first give a short review on a “curvature symmetry” of a semi-Riemannian manifold and then design two mathematical models which link this symmetry with a class of ARS-manifolds (M, g, V, Φ) . Also, we show that there exists a third model of non-Einstein conformally recurrent RS-manifolds. Furthermore, we study a special case when (M, g, V) is conformally flat steady Einstein RS-manifold, supported by a specific example. As an application, in Section 3, we have made some progress in presenting a physical model of perfect fluid ARS-spacetimes which also admit a curvature symmetry for all three classes of ARS-solutions. Then, we prove the existence of a family of ARS Einstein hypersurfaces of a generalized Robertson-Walker(GRW) spacetime introduced by Alias et al.[3] and propose two open problems.

2. Mathematical models of ARS-manifolds

Recall that in 1993 Duggal[10] introduced a general concept of *symmetry inheritance* on a semi-Riemannian manifold (M, g) , corresponding to a vector field V , if $\mathcal{L}_V \Omega = \alpha \Omega$ for some scalar field α , where Ω is any geometric/physical quantity. Well-known example is the case when $\Omega = g$ for which V is a *conformal Killing Vector*(CKV) and, in particular, a homothetic vector (HV) or a Killing Vector (KV) according as α is a non-zero constant or zero, respectively. Another case comes from some problems when a semi-Riemannian manifold has not only a Levi-Civita connection ∇ but also a volume form ω and one needs conditions for their compatibility. For a specific example, we know that the divergence of a vector field V is given by $\nabla \cdot V$ and the volume form divergence is given by

$$\mathcal{L}_V \omega = (\text{div}_\omega V)\omega.$$

Then, ∇ and ω are compatible if $\text{div}_\omega = \nabla \cdot V$ and ω inherits a symmetry defined by above equation with respect to the vector field V .

Since the metric and curvature symmetries play important role in mathematics and physics (see Duggal-Sharma[12]), in this paper we prescribe $\Omega = R^i_{jkm}$ the curvature tensor of (M, g) admitting a *curvature inheritance*(CI)symmetry satisfying

$$\mathcal{L}_V R^i_{jkm} = 2\alpha R^i_{jkm}. \tag{2.1}$$

Denote V by a *curvature inheritance vector* (CIV) field. A CI reduces to a well-known and extensively studied symmetry called *curvature collineation* (CC) when $\alpha = 0$. This third order system in the metric g_{ij} means that the Levi-Civita connection ∇ is a Yang-Mills connection while keeping g on M fixed. For example, Derdzinski [9] constructed such a metric on $S^1 \times N$ for any N carrying the Einstein metric with positive scalar curvature, such that the Ricci tensor is not parallel and the metric is conformally flat. Contracting (2.1) implies

$$\mathcal{L}_V R_{ij} = 2\alpha R_{ij}, \tag{2.2}$$

where R_{ij} is the Ricci tensor of M for which we denote V by *Ricci inheritance vector* (RIV) field. RI reduces to *Ricci collineation* (RC) when $\alpha = 0$. Details on above brief is available in Katzin et al.[16]), Duggal[10, 11], Chapter 8 of Duggal-Sharma[12] and more referred therein. In general, we set

$$\mathcal{L}_V g_{ij} = V_{i;j} + V_{j;i} = P_{ij}. \tag{2.3}$$

For a CIV vector V , using (2.1)- (2.3) the following identities will hold:

$$(a) \quad \mathcal{L}_V R^i_j = 2\alpha R^i_j - R^k_j P^i_k, \quad (b) \quad \mathcal{L}_V r = 2\alpha r - r', \quad r' \equiv R^i_j P^j_i. \tag{2.4}$$

Equation (2.3) raises the question of finding possible values for the tensor P_{ij} which represents the change in g , with respect a CIV field V . For this purpose, we recall that any curvature tensor satisfies the identity: $g_{ij} R^i_{kme} + g_{ik} R^i_{jme} = 0$. Taking Lie derivative of this identity with respect to V , using (2.1) and (2.3) we state

Proposition 2.1. *A necessary condition for a vector V to be a CI vector is that the following curvature identity holds:*

$$P_{ij} R^i_{kme} + P_{ik} R^i_{jme} = 0. \tag{2.5}$$

We highlight that since the necessary condition (2.5) for a CI vector V is independent of the function α it is same for any curvature collineation(CC) vector field for which $\mathcal{L}_V R^i_{jkm} = 0$ holds. Later on reader will see that above curvature symmetry is the same for a special case of another symmetry, called *conformal collineation* symmetry [11]. Moreover, in general, this curvature symmetry places no restriction on any specific type of symmetry vector. Let the general solution of above curvature identity be given by

$$\mathcal{L}_V g_{ij} = 2\Phi g_{ij} + K_{ij}, \tag{2.6}$$

where K_{ij} is a second order symmetric tensor and Φ is a function on M . Then, a particular possibility of (2.6), with prescription $K_{ij} = -2R_{ij}$, provides a link between the CI symmetry of a semi-Riemannian manifold and a class of almost Ricci solitons(ARS) manifolds (M, g, V, Φ) . In support of this link we quote following three results which establish the existence of mathematical models of ARS-manifolds having a symmetry vector V that satisfies the curvature identity (2.5).

Model 1. Katzin et al.[17] has proved that the necessary condition for a non-Einstein conformally flat manifold (M, g, V) to admit a Curvature collineation(CC) vector field V is that

$$\mathcal{L}_V g_{ij} = 2\Phi g_{ij} + \Omega R_{ij}, \quad \Omega = \text{nonzero function on } M,$$

where Φ is a function on M . Thus, for $\Omega = -2$ above equation represent the ARS evolution equation (1.5) so (M, g, Φ, V) is a mathematical model of conformally flat ARS-manifolds satisfying the curvature identity (2.5).

Model 2. Recall that an n dimensional semi-Romannian manifold (M, g, V) admits a “conformal collineation” symmetry [11] defined by a vector field V if

$$\mathcal{L}_V \Gamma_{ij}^k = \delta_i^k \Psi_j + \delta_j^k \Psi_i - \bar{g}_{ij} \Psi^k,$$

where Γ_{ij}^k denotes the Christoffel’s symbols, Ψ is a function and $\Psi_j = \partial_j(\Psi)$. V is then called an “Affine Conformal Vector”(ACV) field of M for which the equation (2.6) holds such that K_{ij} is a covariant constant ($\nabla K = 0$) symmetric tensor field. An ACV reduces to a “conformal Killing vector”, briefly denoted by CKV, if K is proportional to g . Thus, an ACV deviates from a CKV field if there exists a second order covariant constant symmetric tensor $K \neq g$. We know that (M, g) is Ricci symmetric if $\nabla_X Ric(Y, Z) = 0, \forall X, Y, Z \in TM$. Levine-Katzin[18] have proved that if a conformally flat manifold (M, g, Φ, V) admits a conformal collineation symmetry, with CKV vector field V , then, $K_{ij} = ag_{ij} + bR_{ij}$ for some constants a and b and M is Ricci symmetric. Thus, for $a = 0$ and $b = -2$ we recover the evolution equation (1.5). Therefore, (M, g, Φ, V) is a mathematical model of Ricci symmetric ARS-manifolds. See details on this model in a recent paper by Duggal[11]. Now we show that a special case of conformal collineation symmetry admits curvature identity (2.5). Indeed, In general, an ACV vector field V admits following curvature identities (proof is similar to curvature identities on CKV [27]).

$$\begin{aligned} \mathcal{L}_V R_{ij}^m &= \delta_i^m \Phi_{k;j} + \delta_j^m \Phi_{i;k} + \Phi_{;i}^m \bar{g}_{jk} + \Phi_{;j}^m \bar{g}_{ik}, \\ \mathcal{L}_V R_{ij} &= \text{div}(\text{grad}\Phi) \bar{g}_{ij} - (n-1)\Phi_{;ij}, \\ \mathcal{L}_V r &= 2n \text{div}(\text{grad}\Phi) - 2\Phi r + 2r. \end{aligned}$$

If $\Phi = \lambda$ a constant then above identities reduces to

$$\mathcal{L}_V R_{ij}^m = 0 = \mathcal{L}_V R_{ij}, \quad \mathcal{L}_V r = 2(1 - \lambda)r.$$

It follows from above equation that for $\Phi = \lambda$ the conformal collineation symmetry is also a CC symmetry which is a subcase of CI symmetry. Therefore, this model also satisfies the curvature identity (2.5) linked with CC symmetry.

Model 3. Grycak[13] has proved that a non-Einstein conformally recurrent manifold (i.e., $\nabla C = d\theta \otimes C$, where C is the conformal curvature tensor and $d\theta$ is an exact recurrent form) which is neither conformally flat nor recurrent, admits a vector field V satisfying the equation (2.6) such that

$$\mathcal{L}_V g_{ij} = 2\lambda g_{ij} + \Omega R_{ij}, \quad \Omega = \text{nonzero function on } M$$

and λ is a constant. This relates with the Ricci soliton equation (1.3) if we take $\Omega = -2$ so (M, g, λ, V) is a model of RS-manifolds. Although Grycak did not discuss any link of his result with a symmetry vector, but, since above relation is a particular case of general solution of the curvature identity (2.5) it is reasonable to assume that this model also has a link with CI symmetry or one of its subcase. For information on conformally recurrent manifolds, we refer Adati-Miyazawa[1].

Although we have three models with specific prescriptions for the unknown tensor K linking with ARS vector field V , it is reasonable to assume that such a link may also hold for a variety of other types of semi-Riemannian manifolds. For this reason, we state following general result to initiate research on deeper study of possible solutions of the curvature identity (2.5):

Theorem 2.1. Let (M, g, V) be an n -dimensional semi-Riemannian manifold admitting the curvature identity (2.5) with respect to a field V whose general solution is given by the equation (2.6) such that its tensor field $K_{ij} = -2R_{ij}$. Then, V is also an ARS vector field. Therefore, (M, g, Φ, V) is an ARS-manifold.

2.1. ARS-Einstein manifolds

In this subsection we study another possibility of the general solution (2.6) of the CI curvature identity by setting $P_{ij} = 2\sigma g_{ij}$ for which V is a conformal Killing vector (CKV). For this case, we know from Yano [27] that following identities hold:

$$(a) \quad \mathcal{L}_V R_{ij} = -(n-2)\sigma_{;ij} - \Delta \sigma g_{ij}, \quad (b) \quad \mathcal{L}_V r = -2\sigma r - 2(n-1) \Delta \sigma, \quad (2.7)$$

where $\Delta \sigma = \text{div}(\text{grad} \sigma)$. By comparing (2.2),(2.4)a with (2.7)a and then using (2.7)b we obtain

$$(a) \quad \sigma_{;ij} = \frac{\alpha}{n-2} \left(\frac{r}{n-1} g_{ij} - 2R_{ij} \right), \quad (b) \quad \Delta \sigma + \frac{\alpha r}{(n-1)} = 0. \quad (2.8)$$

Therefore, for $\mathcal{L}_V g_{ij} = 2\sigma g_{ij}$, (2.7) reduces to

$$(a) \quad \mathcal{L}_V R_{ij} = -(n-2)\sigma_{;ij} + \frac{\alpha r}{(n-1)} g_{ij}, \quad (b) \quad \mathcal{L}_V r = 2(\alpha - \sigma)r. \quad (2.9)$$

Proposition 2.2. *Under the hypothesis of Theorem 2.1, if the ARS-manifold (M, g, V, Φ) is Einstein ($n > 2$) ($R_{ij} = \frac{r}{n} g_{ij}$, $r \neq 0$), then, V is CKV with conformal function $\Phi = \alpha + \frac{r}{n}$ and M is conformally flat. If $\alpha = 0$, then, (M, g, V) is steady ($\mathcal{L}_V g = 0$) RS-Einstein manifold.*

Proof. Substituting $R_{ij} = \frac{r}{n} g_{ij}$ in the evolution equation (1.5) and using (2.2) with r constant we get

$$(a) \quad \mathcal{L}_V g_{ij} = 2\alpha g_{ij}, \quad (b) \quad \mathcal{L}_V g_{ij} = 2\left(\Phi - \frac{r}{n}\right)g_{ij}. \quad (2.10)$$

Therefore, V is CKV with conformal function $\Phi = \alpha + \frac{r}{n}$ and we know from Yano[27] that M is conformally flat ($\mathcal{L}_V C_{jkm}^i = 0$) where C_{jkm}^i is its conformal curvature tensor. If $\alpha = 0$, then from (2.10) we get $\mathcal{L}_V g_{ij} = 0$ and $\Phi = \frac{r}{n}$ is constant. Thus, (M, g, V) is a steady RS-Einstein manifold which completes the proof. \square

For this case we have following example: Let (M, g, V) be an ARS-Einstein manifold ($R_{ij} = \frac{r}{n} g_{ij}$, $n > 2$) with CKV field V and conformal function $\Phi = \alpha + \frac{r}{n}$. Then, since r is constant we have $\Phi_{;ij} = \alpha_{;ij}$. Taking $R_{ij} = \frac{r}{n} g_{ij}$ in (2.8)b and $\sigma = \alpha$ for r constant, we get

$$\alpha_{;ij} = \rho g_{ij}, \quad \rho \equiv \frac{-r\alpha}{n(n-1)}.$$

Petrov [22] has quoted a result of Sinyukov (1957) that if M admits a vector field α_i satisfying above for a nonzero scalar function ρ , then a system of coordinates exists in which the metric takes the form

$$ds^2 = g_{11} (dx^1)^2 + (g_{11})^{-1} A_{pq}(x^2, \dots, x^n) dx^p dx^q,$$

where $p, q \neq 1$, $g_{11} = [2 \int \rho dx^1 + c]^{-1}$ and $p = p(x^1)$. Thus, this example of M with above metric will hold for the Proposition 2.3.

3. Physical applications

In support of Theorem 2.1, we discuss some physical applications of a class of ARS-spacetimes of relativity. We first examine how the Einstein field equations have effect on the evolution equation (1.5) of an ARS-spacetime. Let

$$G_{ij} \equiv R_{ij} - \frac{1}{2} r g_{ij} = T_{ij}, \quad r = -T_i^i \quad (3.1)$$

be the Einstein field equations where G_{ij} is the Einstein tensor. Suppose (M, g, V) is an ARS-spacetime which satisfies the hypothesis of Theorems 2.1 and admits above Einstein equations. The invariance ($\mathcal{L}_V G = 0$) of G is physically desirable since that amounts to invariance of matter tensor T , useful in finding exact solutions. For example, we know that if a spacetime admits a Killing or homothetic symmetry vector V , then, V leaves G invariant but this invariance property is not obvious for any arbitrary symmetry vector field. An example is the non-invariant conformal Killing vector field (CKV) symmetry. However, we also know that the physically important Robertson-Walker spacetimes do admit nine proper CKV's and six Killing vectors (see, Maartens-Maharaj [19]) and the metric is conformally flat. Here we show that non-Einstein ARS vector field symmetry is also non-invariant but do provides some possible perfect fluid solutions (see Theorem 3.2). With this understanding, we now find condition(s) when $\mathcal{L}_V G = 0$ holds with respect to an ARS symmetry vector field V .

Proposition 3.1. *Let (M, g, V) be an n -dimensional ARS-spacetime ($n > 2$) which admits an ARS-vector field V satisfying Theorem 2.1 and the Einstein field equations (3.1). If V leaves G invariant ($\mathcal{L}_V G = 0$), then, α vanishes and (M, g, V) is conformally flat steady Einstein RS-spacetime.*

Proof. Operating \mathcal{L}_V to both sides of the field equations, using (1.5) and $\mathcal{L}_V T_{ij} = 0$ we obtain $\mathcal{L}_V G_{ij} = 2\alpha R_{ij} - \frac{1}{2} \mathcal{L}_V r g_{ij} - r(\Phi g_{ij} - R_{ij}) = 0$. Therefore, $(2\alpha + r)R_{ij} = (\frac{1}{2} \mathcal{L}_V r + r\Phi)g_{ij}$ so M is Einstein. Now r constant implies that $\mathcal{L}_V r = 0$ so $R_{ij} = \frac{r}{2\alpha+r} \Phi g_{ij}$. Also, we know that M Einstein implies $R_{ij} = \frac{r}{n} g_{ij}$. Comparing these two results we get $n\Phi = 2\alpha + r$. Also, we know from Proposition 2.3 that $n\Phi = n\alpha + r$. Since $n > 2$, the only possibility is that α vanishes and (M, g, V) is conformally flat steady Einstein RS-spacetime which completes the proof. \square

Thus, we conclude that non-Einstein ARS vector field V does not leave G invariant. However, just like the case of proper CKV symmetry we now state and prove a physical model of a perfect fluid ARS-spacetime (M, g, Φ, V) . For this we need following kinematic properties of ARS-spacetimes.

Set $\mathcal{L}_V X^i = \phi X^i + Y^i$, for some function ϕ and non-null unit vector X^i and Y^i where $Y^i X_i = 0$ of M . Contracting this with X_i and then using $\mathcal{L}_V X_i = \mathcal{L}_V g_{ij} X^j$, $\mathcal{L}_V (X_i X^i) = 0$ and the evolution equation (1.5) we obtain

$$\phi = -\epsilon X^i (2\Phi X_i - 2R_{ij} X^j + g_{ij} \mathcal{L}_V X^j).$$

Therefore, $\phi X^i = -2(\Phi + \epsilon R_{ij} X^j X^i) - \mathcal{L}_V X^i$. Substituting this value of ϕX^i in $\mathcal{L}_V X^i = \phi X^i + Y^i$ we get

$$\begin{aligned} (a) \quad \mathcal{L}_V X^i &= -(\Phi + \epsilon R_{jk} X^j X^k) X^i + Y^i/2, \\ (b) \quad \mathcal{L}_V X_i &= (\Phi - \epsilon R_{jk} X^j X^k) X_i - 2R_{ij} X^j + Y_i/2. \end{aligned} \tag{3.2}$$

The set of all integral curves given by a unit non-null or null vector field is called the congruence of non-null or null curves. In this paper, we consider timelike curves, also called flow lines, for which we denote u a unit timelike vector field. For the cases of spacelike and null curves, we refer [12, Chapter 4, pages 57-60]. The acceleration of the flow lines along u is given by $\nabla_u u$. The projection tensor $h_{ij} = g_{ij} + u_i u_j$ is used to project a tangent vector at a point p in M into a spacelike vector orthogonal to u at p . The rate of change of the separation of flow lines from a timelike curve tangent to u is the expansion tensor $\theta_{ij} = h_i^k h_j^m u_{[k;m]}$. The expansion θ , the shear tensor σ_{ij} , the vorticity tensor ω_{ij} and the vorticity vector ω^i are, respectively,

$$\begin{aligned} \theta &= \text{div } u = \theta_{ij} h^{ij}, \quad \sigma_{ij} = \theta_{ij} - \frac{\theta}{n-1} h_{ij}, \\ \omega_{ij} &= h_i^k h_j^m u_{[k;m]}, \quad \omega^i = \frac{1}{2} \eta^{ijkm} u_j \omega_{km}, \end{aligned}$$

where η^{ijkm} is the Levi-Civita volume-form. The covariant derivative of u satisfies

$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \frac{\theta}{n-1} h_{ab} - u_b (u^c_{;a} u_c). \tag{3.3}$$

The rate at which the expansion θ changes along u (with respect to an arc-length parameter s) is given by the Raychaudhuri equation:

$$u\theta = \frac{d\theta}{ds} = -R_{ij} u^i u^j + 2\omega^2 - 2\sigma^2 - \frac{\theta^2}{n-1} + \text{div}(\nabla_u u), \tag{3.4}$$

where $\omega^2 = \frac{1}{2} \omega_{ij} \omega^{ij}$ and $\sigma^2 = \frac{1}{2} \sigma_{ij} \sigma^{ij}$

Theorem 3.1. *Let (M, g, Φ, V) be an n -dimensional ARS-spacetime ($n > 2$) with an ARS-vector field V satisfying Theorem 2.1. Suppose V is parallel to the velocity vector u and (M, g) admits following perfect fluid Einstein field equations*

$$R_{ij} - \frac{1}{2} r g_{ij} = T_{ij} = (\mu + p) u_i u_j + p g_{ij}. \tag{3.5}$$

with μ, p the density and pressure, respectively. Then,

- (a) $(\mu + 3p) = 2(\dot{\beta} - \Phi)$, where $V = \beta u$ and $\beta = -u_i V^i > 0$ is a function. Thus there exists a variety of physically valid solutions for the ARS-spacetimes if and only if $\dot{\beta} > \Phi$.

- (b) For each negative Φ at a point p of the fluid flow there exists a family of expanding ARS-spacetimes for any magnitudes of V .
- (c) For each positive Φ at a point p of the fluid flow the corresponding spacetimes are shrinking if $\dot{\beta} > \Phi$.
- (d) If $\Phi = 0$ then, there exists a variety of physically valid steady solutions of the ARS-spacetimes for any non-constant magnitude of V .

Proof. Using $r = -T_i^i = \mu - 3p$ in (3.5) we get $R_{ij}u^i u^j = \frac{\mu+3p}{2}$. Set in (3.2) $X^i = u^i$ the unit timelike ($\epsilon = -1$) velocity vector. Then, we obtain

$$\begin{aligned} (a) \quad \mathcal{L}_V u^i &= -(\Phi + R_{jk}u^j u^k)u^i + Y^i/2, \\ (b) \quad \mathcal{L}_V u_i &= (\Phi - R_{jk}u^j u^k)u_i - 2R_{ij}u^i + Y_i/2. \end{aligned} \tag{3.6}$$

In general, let $V^i = \beta u^i + \nu^i$ where $\beta = -u_i V^i$ and $\nu^i u_i = 0$. Then, using (3.3) (see details given in [20]) we obtain $\mathcal{L}_V u_i = \dot{\beta}u_i + \beta (\dot{u}_i + (\ln \beta^{-1})_{;j} h_i^j) + \nu^j \dot{u}_j u_i + 2\omega_{ij} \nu^j$. Using this along with the equation (3.6)b and contracting with u^i and $h^{ik} = g^{ik} + u^i u^k$ we get

$$\begin{aligned} (a) \quad \Phi &= \dot{\beta} + \dot{u}_i V^i - R_{ij}u^i u^j, \\ (b) \quad Y_i &= 2\omega_{ij} V^j + \beta (\dot{u}_i + (\ln \beta^{-1})_{;j} h_i^j) + 2R_{jk}u^j h_i^k. \end{aligned} \tag{3.7}$$

Since ARS-vector V is timelike and parallel to u we have $V = \beta u$. Thus, V maps flow lines into flow lines so $Y^i = 0$. Putting $Y^i = 0$ in (3.6)a and then using (3.7)a we get $(\mu + 3p) = 2(\dot{\beta} - \Phi)$. Now for any physically valid fluid solution the energy conditions demand that $\mu + 3p > 0$. Also, $\dot{\beta} > 0$. Therefore, (a) holds. The cases (b) and (c) follow from (a) and the definition of expanding or shrinking solitons, respectively. For the case (d), if $\Phi = 0$ then, $(\mu + 3p) = 2\dot{\beta}$ is positive for any non-constant magnitude of V , which completes the proof. \square

Remark 3.1. Recall that a material curve in a fluid is a curve whose fluid particles move along the curve as the fluid evolves. Material curves play an important role in relativistic fluids. In our case, the timelike fluid flow curves are material curves as V maps flow lines into flow lines.

3.1. ARS Einstein hypersurfaces of GRW-spacetimes

Suppose (\bar{M}, \bar{g}) is a $(1 + n)$ -dimensional Lorentzian manifold whose metric \bar{g} satisfies the following kinematic condition

$$\bar{\nabla}_{\bar{X}} u = f(\bar{X} + \bar{g}(\bar{X}, u)u), \quad \forall \bar{X} \in T\bar{M}, \tag{3.8}$$

where u is a timelike unit vector, $\bar{\nabla}$ denotes the Levi-Civita connection and f is a function on \bar{M} . Consider Gaussian normal coordinates $(x^0 = t, x^a)$ on \bar{M} such that $u = \partial/\partial t$ and $\bar{g} = \bar{g}_{ij} dx^i dx^j = -dt^2 + \bar{g}_{ab} dx^a dx^b$, where i, j are over $0, 1, \dots, n$ and a, b over $1, \dots, n$ and \bar{g}_{ab} are functions of all the coordinates x^a . Let $M_{t(t=q)}$, for some constant q , be a spacelike slice orthogonal to u and $X_t = \partial/\partial x^a$ a coordinate vector field tangent to M_t . Then equation (3.8) reduces to $\bar{\nabla}_{\partial/\partial x^a} \partial/\partial t = f(\partial/\partial x^a)$ which further implies (with some computation) that

$$\frac{\partial \bar{g}_{ab}}{\partial t} = 2f \bar{g}_{ab}. \tag{3.9}$$

Integrating (3.1) we obtain $\bar{g}_{ab} = e^{2 \int f(t, x^a) dt} \gamma_{ab}$, where γ_{ab} is a fixed Riemannian metric on the initial slice $t = 0$. Set $e^{\int f(t, x^a) dt} = S(t, x^a)$. Then, the metric \bar{g} of (\bar{M}, \bar{g}) is of the form $\bar{g}_{ij} = -dt^2 + S^2(t, x^a) \gamma_{ab} dx^a dx^b$ whose spacelike slices orthogonal to u are self-conformal. If S is a function of t alone, then, above metric reduces to the following Generalized Robertson-Walker (GRW) spacetime [3]:

$$\bar{g}_{ij} = -dt^2 + S^2(t) \gamma_{ab} dx^a dx^b \tag{3.10}$$

which we assume in this paper. It is a warped product, with base an open interval of a real line of negatively defined metric and fibre a Riemannian manifold not of constant sectional curvature, in general. The reader will see that the kinematic condition (3.8) on the spacetime metric plays an important role in physical use of the Subsection 2.1. Let (M_0, γ) be a fixed spacelike hypersurface in \bar{M} passing through a point p of \bar{M} via the normal exponential map along M_0 in \bar{M} with metric γ induced from \bar{g} . Assume that $x = (x_1, \dots, x_n)$ are the coordinates in (M_t, g_t) centered on p so that the metric g_t is given by

$$g_t = S^2(t) \gamma_{ab} dx^a dx^b, \quad 1 \leq a, b \leq n.$$

Denote by $[g_t]$ and $[u_t]$ the families of Riemannian metrics and unit normals, respectively, where each u_t satisfies the relation (3.8) and

$$F = \{[M_t], [g_t], [u_t] : [g_t] = [S^2(t)]\gamma\} \tag{3.11}$$

a family of spacelike hypersurfaces of (\bar{M}, \bar{g}) conformally related to the spacelike hypersurface (M_0, γ, u_0) . We take (M_t, g_t, u_t) a member of the family F for some $t = q$, with the understanding that the results are same for any other member, where g is its induced Riemannian metric of the form

$$g_t(X_t, Y_t) = S^2(t)\gamma(X_t, Y_t), \quad \forall X_t, Y_t \in TM_t. \tag{3.12}$$

The Gauss-Weingarten formulas are

$$\begin{aligned} \bar{\nabla}_{X_t} Y_t &= \nabla_{X_t} Y_t + B(X_t, Y_t)u_t \\ \bar{\nabla}_{X_t} u_t &= A_{u_t} X_t, \quad \forall X_t, Y_t \in TM_t, \end{aligned}$$

where $B(X_t, Y_t) = g(A_{u_t} X_t, Y_t)$ and A_{u_t} denotes shape operator. From these two equations we obtain

$$\begin{aligned} \bar{R}(X_t, Y_t)Z_t &= R(X_t, Y_t)Z_t + A_{B(Y_t, Z_t)u_t} X_t - A_{B(X_t, Z_t)u_t} Y_t \\ &\quad + (\nabla_{X_t} B)(Y_t, Z_t) - (\nabla_{Y_t} B)(X_t, Z_t), \end{aligned} \tag{3.13}$$

where \bar{R} and R denote the curvature tensors of \bar{M} and M_t , respectively and $X(u)$ vanishes $\forall X \in TM_t$. Also,

$$(\nabla_{X_t} B)(Y_t, Z_t) = X_t(B(Y_t, Z_t)) - B(\nabla_{X_t} Y_t, Z_t) - B(Y_t, \nabla_{X_t} Z_t).$$

Theorem 3.2. *Let (\bar{M}, \bar{g}) be a $(1 + n)$ -dimensional GRW-spacetime such that \bar{g} satisfies (3.8) and $F = \{[M_t], [g_t], [u_t] : [g_t] = [S^2(t)]\gamma\}$ is a family of its spacelike hypersurfaces. Then,*

- (1) *each member (M_t, g_t, u_t) of F is totally umbilical in \bar{M} with its mean curvature equal to $f = \text{Trace}(A_{u_t})/n$.*
- (2) *If f is a non-zero constant and \bar{M} is Ricci flat then each member of F is an Einstein hypersurface whose non-zero scalar curvature $r = 2(nf)^2$.*
- (3) *If $f = -(r/n)$ and $n > 2$, then, each member of F is ARS-Einstein hypersurface with $r = 1/2$ and the evolution equation (1.5) is given by*

$$\mathcal{L}_V g_t(X_t, Y_t) = 2(\Phi - (1/2n))g_t(X_t, Y_t) \quad \forall X_t, Y_t \in TM_t. \tag{3.14}$$

Proof. Using (3.8) in the Weingarten equation we get $A_{u_t} X_t = fX_t$ which implies $B(X_t, Y_t) = fg(X_t, Y_t)$. Therefore, each M_t is totally umbilical in \bar{M} with $f = \text{Tr}(A_{u_t})/n$ which proves (1). Now using $A_{u_t} X_t = fX_t$, $B(X_t, Y_t) = fg(X_t, Y_t)$ and f constant we get

$$\begin{aligned} A_{B(Y_t, Z_t)u_t} X_t &= fg(Y_t, Z_t)A_{u_t} X_t = f^2g(Y_t, Z_t)X_t, \\ A_{B(X_t, Z_t)u_t} Y_t &= fg(X_t, Z_t)A_{u_t} Y_t = f^2g(X_t, Z_t)Y_t. \\ \bar{R}(X_t, Y_t)Z_t &= R(X_t, Y_t)Z_t - f^2(g(X_t, Z_t)Y_t - g(Y_t, Z_t)X_t) + 0 - 0. \end{aligned}$$

Then, (3.13) reduces to

$$\bar{R}(X_t, Y_t)Z_t = R(X_t, Y_t)Z_t - 2f^2g(X_t, Z_t)Y_t,$$

where we have used one of the curvature identities. Thus, we get the following relation between the Ricci tensors of \bar{M} and M_t .

$$\bar{R}(X_t, Y_t) = R(X_t, Y_t) - 2nf^2g(X_t, Y_t), \quad \forall X_t, Y_t \in TM_t.$$

By hypothesis, $\bar{R}(X_t, Y_t) = 0$. Therefore, $R(X_t, Y_t) = 2nf^2g(X_t, Y_t)$ which implies that (M_t, g_t) is an Einstein hypersurface whose non-zero scalar curvature $r = 2(nf)^2$ which proves (2). Now let $f = -r/n$. Substituting $A_{u_t} X_t = fX_t$ for $X_t = \partial/\partial x^a$ in the Weingarten equation and with straightforward computation we get

$$\frac{\partial g_t}{\partial t} = -(2r/n)g_t,$$

which is the Ricci flow equation (1.1) for the Einstein $(Ric = (r/n)g_t, r \neq 0, (n > 2))$ hypersurface (M_t, g_t) . Moreover, taking $f = -(r/n)$ in $r = 2(nf)^2$ we get $r = 2r^2$. Since $r \neq 0$ the only possibility is that $r = 1/2$. Thus, (M_t, g_t) is ARS-Einstein hypersurface (see details in Subsection 2.1) which completes the proof. \square

The proof of following corollary easily follows from the Proposition 2.2.

Corollary 3.1. *Under the hypothesis of Theorem 3.2, if $f = -(1/n)$ then each (M_t, g_t) admits RI symmetry $(\mathcal{L}_V R(X_t, Y_t) = 2\alpha R(X_t, Y_t))$ with $\alpha = \Phi - 1/2n$. If $\alpha = 0$, then, each (M_t, g_t) is steady RS-Einstein hypersurface of (\bar{M}, \bar{g}) .*

4. Discussions

In this paper we have used some basic results of a curvature inheritance(CI) symmetry in the study of semi-Riemannian ARS-manifolds and their applications to spacetimes of general relativity. The motivation originated from the observation that the evolution equation (1.5) of the ARS vector V is a particular case of the general solution (2.6) of the curvature identity (2.5) of CI symmetry. This work is the sequel to my previous paper [11] which was a restricted case of covariant constant Ricci tensor of an ARS-manifolds. Here we claim that use of a CI symmetry in this paper is an important step forward towards an improvement of our previous paper on the geometry and physics of almost Ricci solitons. To clarify our claim, we first show that our Theorem 2.1 (supported by three mathematical models of ARS-manifolds) is applicable to a large variety of Riemannian and semi-Riemannian ARS-manifolds and not just the restricted case of Ricci symmetric manifolds. On the physical use of our Theorem 2.1, recall from a paper by Hall and Da-Costa [14] that the existence of covariant constant Ricci tensor must exclude some spacetimes including the case of perfect fluid Einstein field equations. We highlight that contrary to the previous restricted paper our Theorem 3.1 shows the possible existence of perfect fluid solutions for all the three cases of ARS-spacetimes. Also, notice from the three classes of solutions of Theorem 3.1 that as the fluid revolves parallel to the velocity vector u , the deviation of the Ricci tensor R_{ij} from the metric tensor g_{ij} causes change in the fluid pattern which is governed by the two variables β and Φ . Therefore, our physical model is applicable to any specific problem under investigation by adjusting the magnitude of the ARS vector field V . Moreover, our Theorem 3.2 opens the possibility of study on mean curvature flow of ARS-hypersurfaces of semi-Riemannian or spacetime manifolds.

Open problems.(a) Observe that Theorem 3.1 only gives general relations for $(\mu + 3p)$ in terms of the two quantities (β, Φ) for all the three cases (expanding, steady or shrinking) but, just like the case of examples for the Riemannian Ricci solitons discussed in Chow-Knopf [6, Chapter 2], there is a need to find specific examples of solutions for the three cases. Thus, we propose further research on

“ Exact perfect fluid solutions for the three cases of the ARS-spacetimes”.

(b) Since the modified theory of the almost Ricci solitons(ARS) has just been introduced, there is a need to develop its basic results with focus on similarities and differences with the key results of Riemannian Ricci solitons and to justify the replacing of constant λ by a function Φ in the ARS evolution equation. A complete analysis on the existence of three classes of ARS-solutions for Riemannian and semi-Riemannian manifolds with their physical interpretation is desirable.

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