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On Biharmonic Lorentz Hypersurfaces with Non-Diagonal Shape Operator

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ABSTRACT

We prove that there exist no proper biharmonic Lorentz hypersurface M_1^n in E_1^{n+1} with at most three distinct principal curvatures of non-diagonal shape operator having minimal polynomial $(y-\lambda)^2(y-\lambda_1)(y-\lambda_n)$.

Keywords: *Pseudo-Euclidean space; Biharmonic submanifolds; Mean curvature vector.*

AMS Subject Classification (2010): *Primary: 53D12 ; Secondary: 53C40; 53C42.*

1. Introduction

Let M_r^n be an n-dimensional, connected submanifold of the pseudo-Euclidean space E_s^m. Denote by $\vec{x}$ and $\triangle$ respectively the position vector field and the Laplace operator on M_r^n with respect to the induced metric g on M_r^n, from the indefinite metric on the ambient space E_s^m. It is well known that

$$\triangle\vec{x} = -n\vec{H},$$

where $\vec{H}$ is the mean curvature vector of M. An immersion is minimal ($\vec{H}=0$) if and only if $\triangle\vec{x}=0$ and is called biharmornic if $\triangle^2\vec{x}=0$ i.e. $\triangle\vec{H}=0$. Of course, for an immersion, minimality implies biharmonicity.

The study of submanifolds with harmonic mean curvature vector field was initiated by Chen in 1985 and arose in the context of his theory of submanifolds of finite type. A survey on submanifolds of finite type and various related topics was presented in [4, 5].

In 1991, Chen conjectured the following:

Conjecture: *The only biharmonic submanifolds of Euclidean spaces are the minimal ones.*

In Euclidean spaces, we have the following results, which indeed support the above mentioned conjecture. Chen proved in 1985 that every biharmonic surface in E^3 is minimal. Thereafter, I. Dimitric generalized this result [9]. In [14], it was proved that every biharmonic hypersurface in E^4 is minimal. In [16], it was obtained that every biharmonic hypersurface in E^5 with three distinct principal curvatures must be minimal. Also, it was proved that every biharmonic hypersurfaces with three distinct principal curvatures in E^{n+1} with arbitrary dimension is minimal [12]. Recently, it was proved that there exist no proper biharmonic hypersurfaces in E^5 with zero scalar curvature [10].

Chen et al. [7, 8] obtained some examples of proper biharmonic surfaces in 4-dimensional pseudo-Euclidean spaces E_s^4 for $s=1,2,3$ (see also [6]). Also, it was proved in [7, 8] that biharmonic surfaces in pseudo-Euclidean 3-spaces are minimal. A. Arvanitoyeorgos et al. [2] proved that biharmonic Lorentzian hypersurfaces in Minkowski 4-spaces are minimal. In [16], it was proved that every biharmonic non-degenerate hypersurface in E_s^5 with three distinct principal curvatures of diagonal shape operator is minimal.

In this paper, we study biharmonic Lorentz hypersurfaces M_1^n in E_1^{n+1} with at most three distinct eigen values of non-diagonal shape operators satisfies the equation (2.11).

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2. Preliminaries

Let (M_1^n, g) be a n-dimensional Lorentz hypersurface isometrically immersed in a $n+1$-dimensional pseudo-Euclidean space $(E_1^{n+1}, \overline{g})$ and $g = \overline{g}_{|M_1^n}$. We denote by ξ unit normal vector to M_1^n with $\overline{g}(\xi, \xi) = 1$.

Let $\overline{\nabla}$ and ∇ denote linear connections on E_1^{n+1} and M_1^n, respectively. Then, the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \ \ \forall \ \ X, Y \in \Gamma(TM_1^n), \tag{2.1}$$

$$\overline{\nabla}_X \xi = -S_\xi X, \ \ \forall \ \ \xi \in \Gamma(TM_1^n)^\perp, \tag{2.2}$$

where h is the second fundamental form and S is the shape operator. It is well known that the second fundamental form h and shape operator S are related by

$$\overline{g}(h(X, Y), \xi) = g(S_\xi X, Y). \tag{2.3}$$

The mean curvature vector is given by

$$\vec{H} = \frac{1}{n}\text{trace}h. \tag{2.4}$$

The Gauss and Codazzi equations are given by

$$R(X, Y)Z = g(SY, Z)SX - g(SX, Z)SY, \tag{2.5}$$

$$(\nabla_X S)Y = (\nabla_Y S)X, \tag{2.6}$$

respectively, where R is the curvature tensor, $S = S_\xi$ for some unit normal vector field ξ and

$$(\nabla_X S)Y = \nabla_X(SY) - S(\nabla_X Y), \tag{2.7}$$

for all $X, Y, Z \in \Gamma(TM_1^n)$.

By comparing the tangential and normal components in biharmonic equation $\triangle\vec{H} = 0$, the necessary and sufficient conditions for M_1^n to have proper mean curvature in E_1^{n+1} are

$$\triangle H + H\text{trace}S^2 = 0, \tag{2.8}$$

and

$$S(\text{grad}H) + \frac{n}{2}H\text{grad}H = 0, \tag{2.9}$$

where H denotes the mean curvature. Also, the Laplace operator $\triangle$ of a scalar valued function f is given by [3]

$$\triangle f = -\sum_{i=1}^{n} \epsilon_i(e_i e_i f - \nabla_{e_i} e_i f), \tag{2.10}$$

where $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal local tangent frame on M_1^n with $\epsilon_i = \pm 1$.

A vector X in E_s^{n+1} is called spacelike, timelike or lightlike according as $\overline{g}(X, X) > 0$, $\overline{g}(X, X) < 0$ or $\overline{g}(X, X) = 0$, respectively. A non-degenerate hypersurface M_r^n of E_s^{n+1} is called Riemannian or pseudo-Riemannian according as the induced metric on M_r^n from the indefinite metric on E_s^{n+1} is definite or indefinite. A shape operator of pseudo-Riemannian hypersurfaces is not diagonalizable always unlike the Riemannian hypersurfaces.

It was proved in [16, 15] that the canonical form of the non-diagonal shape operator of M_1^n in E_1^{n+1} having minimal polynomial $(y-\lambda)^2(y-\lambda_1)(y-\lambda_n)$ with three distinct real eigen values takes the form

$$S = \begin{pmatrix} \lambda & 0 & & & & & & & 0 \\ 1 & \lambda & & & & & & & \\ & & \lambda & & & & & & \\ & & & \ldots & & & & & \\ & & & & \lambda_1 & & & & \\ & & & & & \ldots & & & \\ & & & & & & \lambda_n & & \\ & & & & & & & \ldots & \\ 0 & & & & & & & & \lambda_n \end{pmatrix}, \tag{2.11}$$

with respect to some suitable pseudo-orthonormal frame of the tangent bundle.

3. Biharmonic Lorentz hypersurfaces in E_1^{n+1} with non-diagonal shape operator

Let M_1^n be a biharmonic Lorentz hypersurface in E_1^{n+1} with proper mean curvature vector field having non-diagonal shape operator given by (2.11). Also, we assume that mean curvature is not constant and $\text{grad}H \neq 0$. Assuming non constant mean curvature implies the existence of an open connected subset U of M_1^n, with $\text{grad}_p H \neq 0$, for all $p \in U$. The shape operator S of a biharmonic Lorentz hypersurface given by (2.11) having the three distinct eigen values λ, λ_1 and λ_n with multiplicities r, s and t respectively, and with minimal polynomial $(y-\lambda)^2(y-\lambda_1)(y-\lambda_n)$ can be written as

$$S(e_1) = \lambda e_1 + e_2, \quad S(e_2) = \lambda e_2, \quad S(e_A) = \lambda e_A, \quad S(e_B) = \lambda_1 e_B, \quad S(e_C) = \lambda_n e_C, \tag{3.1}$$

with respect to pseudo orthonormal basis of vector fields $\{e_1, e_2, \ldots, e_n\}$ of $T_pM_1^n$, satisfying

$$g(e_1, e_2) = -1, \quad g(e_i, e_i) = 1, \tag{3.2}$$

and

$$g(e_1, e_1) = g(e_2, e_2) = g(e_1, e_i) = g(e_2, e_i) = g(e_i, e_j) = 0, \tag{3.3}$$

for $i \neq j$ and $i, j = 3, 4, \ldots, n,$ and $A = 3, 4, \ldots, r, \quad B = r+1, r+2, \ldots, r+s, \quad C = r+s+1, r+s+2, \ldots, r+s+t = n.$

We write

$$\nabla_{e_\beta} e_\gamma = \sum_{k=1}^{n} \omega_{\beta\gamma}^k e_k, \qquad \beta, \gamma = 1, 2, 3, \ldots, n. \tag{3.4}$$

Taking covariant derivatives of (3.2) and (3.3) with respect to e_k and using (3.4), we find

$$\omega_{k1}^1 = -\omega_{k2}^2, \quad \omega_{ki}^i = \omega_{k1}^2 = \omega_{k2}^1 = 0, \quad \omega_{k1}^i = \omega_{ki}^2, \quad \omega_{k2}^i = \omega_{ki}^1, \quad \omega_{ki}^j = -\omega_{kj}^i, \tag{3.5}$$

for $i \neq j, \quad i, j = 3, 4, \ldots, n,$ and $k = 1, 2, \ldots, n.$

Now onwards, we take

$$\begin{aligned} &A \neq \widetilde{A}, \quad A, \widetilde{A} = 3, 4, \ldots, r, \\ &B \neq \widetilde{B}, \quad B, \widetilde{B} = r+1, r+2, \ldots, r+s, \\ &C \neq \widetilde{C}, \quad C, \widetilde{C} = r+s+1, r+s+2, \ldots, r+s+t = n. \end{aligned}$$

Putting $X = e_1, Y = e_2$ in (2.6), and using (2.7) and (3.1), gives

$$\begin{aligned} &e_1(\lambda)e_2 + \lambda \textstyle\sum_{p\neq 1} \omega_{12}^p e_p - \omega_{12}^2(\lambda e_2) - \sum_{A=3}^{r} \omega_{12}^A(\lambda e_A) - \sum_{B=r+1}^{r+s} \omega_{12}^B(\lambda_1 e_B) \\ &- \textstyle\sum_{C=r+s+1}^{n} \omega_{12}^C(\lambda_n e_C) = e_2(\lambda)e_1 + \lambda \sum_{p\neq 2} \omega_{21}^p e_p + \sum_{p\neq 1} \omega_{22}^p e_p - \omega_{21}^1(\lambda e_1 + e_2) \\ &- \textstyle\sum_{A=3}^{r} \omega_{21}^A(\lambda e_A) - \sum_{B=r+1}^{r+s} \omega_{21}^B(\lambda_1 e_B) - \sum_{C=r+s+1}^{n} \omega_{21}^C(\lambda_n e_C), \end{aligned}$$

whereby, taking inner product with e_2, e_A, we obtain

$$e_2(\lambda) = 0, \tag{3.6}$$

and

$$\omega_{22}^A = 0, \tag{3.7}$$

respectively.

Putting $X = e_1, Y = e_B$ in (2.6), and using (2.7) and (3.1), gives

$$\begin{aligned} &e_1(\lambda_1)e_B + \lambda_1 \textstyle\sum_{p\neq B} \omega_{1B}^p e_p - \omega_{1B}^1(\lambda e_1 + e_2) - \omega_{1B}^2(\lambda e_2) - \sum_{A=3}^{r} \omega_{1B}^A(\lambda e_A) - \\ &\textstyle\sum_{\widetilde{B}=r+1}^{r+s} \omega_{1B}^{\widetilde{B}}(\lambda_1 e_{\widetilde{B}}) - \sum_{C=r+s+1}^{n} \omega_{1B}^C(\lambda_n e_C) = e_B(\lambda)e_1 + \lambda \sum_{p\neq 2} \omega_{B1}^p e_p + \sum_{p\neq 1} \omega_{B2}^p e_p \\ &-\omega_{B1}^1(\lambda e_1 + e_2) - \textstyle\sum_{A=3}^{r} \omega_{B1}^A(\lambda e_A) - \sum_{m=r+1}^{r+s} \omega_{B1}^m(\lambda_1 e_m) - \sum_{C=r+s+1}^{n} \omega_{B1}^C(\lambda_n e_C), \end{aligned}$$

whereby, taking inner product with e_2, e_A, $e_{\widetilde{B}}$ and e_C, we get

$$e_B(\lambda) = (\lambda_1 - \lambda)\omega_{1B}^1, \tag{3.8}$$

$$(\lambda_1 - \lambda)\omega_{1B}^A = \omega_{B2}^A, \tag{3.9}$$

$$(\lambda - \lambda_1)\omega_{B1}^{\widetilde{B}} + \omega_{B2}^{\widetilde{B}} = 0, \tag{3.10}$$

and

$$(\lambda_1 - \lambda_n)\omega_{1B}^C = (\lambda - \lambda_n)\omega_{B1}^C + \omega_{B2}^C, \tag{3.11}$$

respectively.

Putting $X = e_1$, $Y = e_C$ in (2.6), and using (2.7) and (3.1), gives

$$e_1(\lambda_n)e_C + \lambda_1 \sum_{p\neq C} \omega_{1C}^p e_p - \omega_{1C}^1(\lambda e_1 + e_2) - \omega_{1C}^2(\lambda e_2) - \sum_{A=3}^r \omega_{1C}^A(\lambda e_A) - \sum_{B=r+1}^{r+s} \omega_{1C}^B(\lambda_1 e_B) - \sum_{\widetilde{C}=r+s+1}^n \omega_{1C}^{\widetilde{C}}(\lambda_n e_{\widetilde{C}}) = e_C(\lambda)e_1 + \lambda \sum_{p\neq 2} \omega_{C1}^p e_p + \sum_{p\neq 1} \omega_{C2}^p e_p - \omega_{C1}^1(\lambda e_1 + e_2) - \sum_{A=3}^r \omega_{C1}^A(\lambda e_A) - \sum_{B=r+1}^{r+s} \omega_{C1}^B(\lambda_1 e_B) - \sum_{m=r+s+1}^n \omega_{C1}^m(\lambda_n e_m),$$

whereby, taking inner product with e_2, e_A, and e_C, we have

$$e_C(\lambda) = (\lambda_n - \lambda)\omega_{1C}^1, \tag{3.12}$$

$$(\lambda_n - \lambda)\omega_{1C}^A = \omega_{C2}^A, \tag{3.13}$$

and

$$e_1(\lambda_n) = (\lambda - \lambda_n)\omega_{C1}^C + \omega_{C2}^C, \tag{3.14}$$

respectively.

Putting $X = e_2$, $Y = e_B$ in (2.6) and using (2.7) and (3.1), we get

$$e_2(\lambda_1)e_B + \lambda_1 \sum_{p\neq B} \omega_{2B}^p e_p - \omega_{2B}^1(\lambda e_1 + e_2) - \omega_{2B}^2(\lambda e_2) - \sum_{A=3}^r \omega_{2B}^A(\lambda e_A) - \sum_{\widetilde{B}=r+1}^{r+s} \omega_{2B}^{\widetilde{B}}(\lambda_1 e_{\widetilde{B}}) - \sum_{C=r+s+1}^n \omega_{2B}^C(\lambda_n e_C) = e_B(\lambda)e_2 + \lambda \sum_{p\neq 1} \omega_{B2}^p e_p - \omega_{B2}^2(\lambda e_2) - \sum_{A=3}^r \omega_{B2}^A(\lambda e_A) - \sum_{m=r+1}^{r+s} \omega_{B2}^m(\lambda_1 e_m) - \sum_{C=r+s+1}^n \omega_{B2}^C(\lambda_n e_C),$$

whereby, taking inner product with e_1, e_2, e_A, e_B, $e_{\widetilde{B}}$, and e_C, we find

$$e_B(\lambda) = (\lambda_1 - \lambda)\omega_{2B}^2 - \omega_{2B}^1, \tag{3.15}$$

$$\omega_{2B}^1 = 0, \tag{3.16}$$

$$\omega_{2B}^A = 0, \tag{3.17}$$

$$(\lambda - \lambda_1)\omega_{B2}^B = e_2(\lambda_1), \tag{3.18}$$

$$\omega_{B2}^{\widetilde{B}} = 0, \tag{3.19}$$

and

$$(\lambda_1 - \lambda_n)\omega_{2B}^C = (\lambda - \lambda_n)\omega_{B2}^C, \tag{3.20}$$

respectively.

Putting $X = e_2$, $Y = e_C$ in (2.6), and using (2.7) and (3.1), gives

$$e_2(\lambda_n)e_C + \lambda_n \sum_{p\neq C} \omega_{2C}^p e_p - \omega_{2C}^1(\lambda e_1 + e_2) - \omega_{2C}^2(\lambda e_2) - \sum_{A=3}^r \omega_{2C}^A(\lambda e_A) - \sum_{B=r+1}^{r+s} \omega_{2C}^B(\lambda_1 e_B) - \sum_{\widetilde{C}=r+s+1}^n \omega_{2C}^{\widetilde{C}}(\lambda_n e_{\widetilde{C}}) = e_C(\lambda)e_2 + \lambda \sum_{p\neq 1} \omega_{C2}^p e_p - \omega_{C2}^2(\lambda e_2) - \sum_{A=3}^r \omega_{C2}^A(\lambda e_A) - \sum_{B=r+1}^{r+s} \omega_{C2}^B(\lambda_1 e_B) - \sum_{m=r+s+1}^n \omega_{C2}^m(\lambda_n e_m),$$

whereby, taking inner product with e_1, e_2, e_A, e_C and $e_{\widetilde{C}}$, we obtain

$$(\lambda_n - \lambda)\omega_{2C}^2 = e_C(\lambda) + \omega_{2C}^1, \tag{3.21}$$

$$\omega_{2C}^1 = 0, \tag{3.22}$$

$$\omega_{2C}^A = 0, \tag{3.23}$$

$$(\lambda - \lambda_n)\omega_{C2}^C = e_2(\lambda_n), \tag{3.24}$$

and

$$\omega_{C2}^{\widetilde{C}} = 0, \tag{3.25}$$

respectively.

Putting $X = e_A$, $Y = e_B$ in (2.6), and using (2.7) and (3.1), gives

$$e_A(\lambda_1)e_B + \lambda_1 \sum_{p\neq B} \omega_{AB}^p e_p - \omega_{AB}^1(\lambda e_1 + e_2) - \omega_{AB}^2(\lambda e_2) - \sum_{m=3}^r \omega_{AB}^m(\lambda e_m) - \sum_{\widetilde{B}=r+1}^{r+s} \omega_{AB}^{\widetilde{B}}(\lambda_1 e_{\widetilde{B}}) - \sum_{C=r+s+1}^n \omega_{AB}^C(\lambda_n e_C) = e_B(\lambda)e_A + \lambda \sum_{p\neq A} \omega_{BA}^p e_p - \omega_{BA}^1 (\lambda e_1 + e_2) - \omega_{BA}^2(\lambda e_2) - \sum_{\widetilde{A}=3}^r \omega_{BA}^{\widetilde{A}}(\lambda e_{\widetilde{A}}) - \sum_{m=r+1}^{r+s} \omega_{BA}^m(\lambda_1 e_m) - \sum_{C=r+s+1}^n \omega_{BA}^C(\lambda_n e_C),$$

whereby, taking inner product with e_2, e_A, $e_{\widetilde{A}}$, e_B, $e_{\widetilde{B}}$, and e_C, we get

$$\omega_{AB}^1 = 0, \tag{3.26}$$

$$(\lambda_1 - \lambda)\omega_{AB}^A = e_B(\lambda), \tag{3.27}$$

$$\omega_{AB}^{\widetilde{A}} = 0, \tag{3.28}$$

$$(\lambda - \lambda_1)\omega_{BA}^B = e_A(\lambda_1), \tag{3.29}$$

$$\omega_{BA}^{\widetilde{B}} = 0, \tag{3.30}$$

and

$$(\lambda_1 - \lambda_n)\omega_{AB}^C = (\lambda - \lambda_n)\omega_{BA}^C, \tag{3.31}$$

respectively.

Putting $X = e_A$, $Y = e_C$ in (2.6), and using (2.7) and (3.1), gives

$$e_A(\lambda_n)e_C + \lambda_n \sum_{p\neq C} \omega_{AC}^p e_p - \omega_{AC}^1(\lambda e_1 + e_2) - \omega_{AC}^2(\lambda e_2) - \sum_{m=3}^r \omega_{AC}^m(\lambda e_m) - \sum_{B=r+1}^{r+s} \omega_{AC}^B(\lambda_1 e_B) - \sum_{\widetilde{C}=r+s+1}^n \omega_{AC}^{\widetilde{C}}(\lambda_n e_{\widetilde{C}}) = e_C(\lambda)e_A + \lambda \sum_{p\neq A} \omega_{CA}^p e_p - \omega_{CA}^1 (\lambda e_1 + e_2) - \omega_{CA}^2(\lambda e_2) - \sum_{\widetilde{A}=3}^r \omega_{CA}^{\widetilde{A}}(\lambda e_{\widetilde{A}}) - \sum_{B=r+1}^{r+s} \omega_{CA}^B(\lambda_1 e_B) - \sum_{m=r+s+1}^n \omega_{CA}^m(\lambda_n e_m),$$

whereby, taking inner product with e_2, e_A, $e_{\widetilde{A}}$, e_B and e_C, we find

$$\omega_{AC}^1 = 0, \tag{3.32}$$

$$(\lambda_n - \lambda)\omega_{AC}^A = e_C(\lambda), \tag{3.33}$$

$$\omega_{AC}^{\widetilde{A}} = 0, \tag{3.34}$$

$$(\lambda_n - \lambda_1)\omega_{AC}^B = (\lambda - \lambda_1)\omega_{CA}^B, \tag{3.35}$$

and

$$(\lambda - \lambda_n)\omega_{CA}^C = e_A(\lambda_n), \tag{3.36}$$

respectively.

Putting $X = e_B$, $Y = e_C$ in (2.6), and using (2.7) and (3.1), gives

$$e_B(\lambda_n)e_C + \lambda_n \sum_{p\neq C} \omega_{BC}^p e_p - \omega_{BC}^1(\lambda e_1 + e_2) - \omega_{BC}^2(\lambda e_2) - \sum_{A=3}^r \omega_{BC}^A(\lambda e_A) - \sum_{m=r+1}^{r+s} \omega_{BC}^m(\lambda_1 e_m) - \sum_{\widetilde{C}=r+s+1}^n \omega_{BC}^{\widetilde{C}}(\lambda_n e_{\widetilde{C}}) = e_C(\lambda_1)e_B + \lambda_1 \sum_{p\neq B} \omega_{CB}^p e_p - \omega_{CB}^1 (\lambda e_1 + e_2) - \omega_{CB}^2(\lambda e_2) - \sum_{A=3}^r \omega_{CB}^A(\lambda e_A) - \sum_{\widetilde{B}=r+1}^{r+s} \omega_{CB}^{\widetilde{B}}(\lambda_1 e_{\widetilde{B}}) - \sum_{m=r+s+1}^n \omega_{CB}^m(\lambda_n e_m),$$

whereby, taking inner product with e_1, e_2, e_B, $e_{\widetilde{B}}$ and e_C, we obtain

$$(\lambda_n - \lambda)\omega_{BC}^2 - \omega_{BC}^1 = (\lambda_1 - \lambda)\omega_{CB}^2 - \omega_{CB}^1, \tag{3.37}$$

$$(\lambda_n - \lambda)\omega_{BC}^1 = (\lambda_1 - \lambda)\omega_{CB}^1, \tag{3.38}$$

$$(\lambda_n - \lambda_1)\omega_{BC}^B = e_C(\lambda_1), \tag{3.39}$$

$$\omega_{BC}^{\widetilde{B}} = 0, \tag{3.40}$$

and

$$(\lambda_1 - \lambda_n)\omega_{CB}^{C} = e_B(\lambda_n), \tag{3.41}$$

respectively.

Similarly, evaluating $g((\nabla_{e_1}S)e_A, e_2) = g((\nabla_{e_A}S)e_1, e_2)$, $g((\nabla_{e_B}S)e_{\widetilde{B}}, e_{\widetilde{B}}) = g((\nabla_{e_{\widetilde{B}}}S)e_B, e_{\widetilde{B}})$, and $g((\nabla_{e_C}S)e_{\widetilde{C}}, e_{\widetilde{C}}) = g((\nabla_{e_{\widetilde{C}}}S)e_C, e_{\widetilde{C}})$, and using (2.7) and (3.1), we get

$$e_A(\lambda) = 0, \tag{3.42}$$

$$e_B(\lambda_1) = 0, \tag{3.43}$$

and

$$e_C(\lambda_n) = 0, \tag{3.44}$$

respectively.

Now, we consider the following cases of grad$H$ viz. space like and light like depending upon preferred direction to study biharmonic Lorentz hypersurfaces in E_1^{n+1} with non-diagonal shape operator given by (2.11). It is obvious from (2.9) that grad$H$ is an eigenvector of the shape operator S with the corresponding eigenvalues $-\frac{nH}{2}$.

Let grad$H$ be light like: Assuming grad$H$ in the direction of e_2, we can write grad$H$= $-e_1(H)e_2$. From (2.9), (2.4) and (3.1), we get

$$\lambda = -\frac{nH}{2} \quad \text{and} \quad \lambda_1 = \frac{nH(n-s-t+2)}{2s} - \frac{t}{s}\lambda_n. \tag{3.45}$$

Since grad$H$= $-e_1(H)e_2$, therefore, using (3.45), we have

$$e_1(H) \neq 0, \quad e_l(H) = 0, \quad e_l(\lambda) = 0, \quad l = 2, 3, \ldots, n. \tag{3.46}$$

Using (3.4), (3.46) and the fact that $[e_l\ e_q](H) = 0 = \nabla_{e_p}e_q(H) - \nabla_{e_q}e_l(H)$, for $l \neq q$ and $l, q = 2, 3, \ldots, n$, we find

$$\omega_{lq}^{1} = \omega_{ql}^{1}. \tag{3.47}$$

First, we consider the case of three distinct eigenvalues viz.

Case I: Let $\lambda - \lambda_1 \neq 0$, $\lambda_n - \lambda_1 \neq 0$ and $\lambda - \lambda_n \neq 0$.

Using (3.26), (3.32), (3.47) and (3.5), we have

$$\omega_{BA}^{1} = \omega_{B2}^{A} = \omega_{CA}^{1} = \omega_{C2}^{A} = 0. \tag{3.48}$$

From (3.7), (3.19), (3.25) and (3.5), we get

$$\omega_{2A}^{1} = \omega_{B\widetilde{B}}^{1} = \omega_{C\widetilde{C}}^{1} = 0. \tag{3.49}$$

Also, using (3.8), (3.12), (3.15), (3.16), (3.21), (3.22), (3.46) and (3.5), we find

$$\omega_{1B}^{1} = \omega_{1C}^{1} = \omega_{2B}^{2} = \omega_{2C}^{2} = 0. \tag{3.50}$$

Using (3.38), (3.47) and (3.5), we obtain

$$\omega_{CB}^{1} = \omega_{BC}^{1} = \omega_{C2}^{B} = \omega_{B2}^{C} = 0. \tag{3.51}$$

Now, from (3.9), (3.13), (3.48), (3.20), (3.51) and (3.5), we have

$$\omega_{1B}^{A} = \omega_{1C}^{A} = \omega_{2B}^{C} = \omega_{2C}^{B} = 0. \tag{3.52}$$

Now, we have the following:

Lemma 3.1. *Let M_1^n be a biharmonic Lorentz hypersurface with non-constant mean curvature in the pseudo Euclidean space E_1^{n+1}, having the non-diagonal shape operator given by (2.11). If grad$H$ is light like and in the direction of e_2, then*

$$\nabla_{e_1} e_B = \sum_{p\neq 1,A,B} \omega_{1B}^p e_p, \nabla_{e_2} e_A = \sum_{p\neq 1,A} \omega_{2A}^p e_p, \quad \nabla_{e_2} e_B = \sum_{p=r+1}^{r+s} \omega_{2B}^p e_p,$$

$$\nabla_{e_2} e_C = \sum_{p=r+s+1}^{n} \omega_{2C}^p e_p, \quad \nabla_{e_A} e_B = \sum_{p\neq 1,B} \omega_{AB}^p e_p, \quad \nabla_{e_A} e_C = \sum_{p\neq 1,C} \omega_{AC}^p e_p, \nabla_{e_B} e_1 = \sum_{p\neq 2} \omega_{B1}^p e_p,$$

$$\nabla_{e_B} e_2 = \sum_{p\neq 1,A,\widetilde{B},C} \omega_{B2}^p e_p, \quad \nabla_{e_B} e_B = \sum_{p\neq B} \omega_{BB}^p e_p, \quad \nabla_{e_B} e_C = \sum_{p\neq 1,C} \omega_{BC}^p e_p,$$

$$\nabla_{e_C} e_1 = \sum_{p\neq 2} \omega_{C1}^p e_p, \quad \nabla_{e_C} e_2 = \sum_{p\neq 1,A,B,\widetilde{C}} \omega_{C2}^p e_p, \quad \nabla_{e_C} e_B = \sum_{p\neq 1,B} \omega_{CB}^p e_p,$$

$$\nabla_{e_C} e_C = \sum_{p\neq C} \omega_{CC}^p e_p, \quad \nabla_{e_B} e_{\widetilde{B}} = \sum_{p\neq 1,\widetilde{B}} \omega_{B\widetilde{B}}^p e_p, \quad \nabla_{e_C} e_{\widetilde{C}} = \sum_{p\neq 1,\widetilde{C}} \omega_{C\widetilde{C}}^p e_p.$$

Now, computing $g(R(e_2, e_B)e_B, e_2)$, $g(R(e_2, e_C)e_C, e_2)$, using (2.5) and Lemma 3.1, we obtain

$$e_2(\omega_{BB}^1) + \omega_{BB}^1(\omega_{21}^1 + \omega_{BB}^1) = 0, \quad e_2(\omega_{CC}^1) + \omega_{CC}^1(\omega_{21}^1 + \omega_{CC}^1) = 0. \tag{3.53}$$

Adding (3.18) and (3.24), and using (3.45), (3.46) and (3.5) therein, we get

$$\{\frac{n(n-t+2)H}{2} - t\lambda_n\}\omega_{BB}^1 + t\{\frac{nH}{2} + \lambda_n\}\omega_{CC}^1 = 0. \tag{3.54}$$

Acting on (3.54) with e_2 and using (3.53), we find

$$2te_2(\lambda_n)[\omega_{CC}^1 - \omega_{BB}^1] = 0,$$

which implies either $e_2(\lambda_n) = 0$ or $\omega_{CC}^1 = \omega_{BB}^1$. In both cases, using (3.18), (3.24), (3.54) and (3.5), we have

$$\omega_{BB}^1 = \omega_{CC}^1 = \omega_{B2}^B = \omega_{C2}^C = 0. \tag{3.55}$$

Now, computing $g(R(e_B, e_1)e_B, e_2)$, $g(R(e_C, e_1)e_C, e_2)$ and using (2.5), Lemma 3.1 and (3.45), we obtain

$$\sum_{A=3}^{r} \omega_{BB}^A \omega_{1A}^1 = -\frac{nH}{2}\{\frac{nH(n-s-t+2)}{2s} - \frac{t}{s}\lambda_n\}, \tag{3.56}$$

and

$$\sum_{A=3}^{r} \omega_{CC}^A \omega_{1A}^1 = -\frac{nH}{2}\lambda_n. \tag{3.57}$$

Now, adding (3.29) and (3.36), and using (3.45), (3.46) and (3.5) therein, we get

$$\{\frac{n(n-t+2)H}{2} - t\lambda_n\}\omega_{BB}^A + t\{\frac{nH}{2} + \lambda_n\}\omega_{CC}^A = 0. \tag{3.58}$$

Since A varies from 3 to r, therefore (3.56), (3.57) and (3.58) is valid for $r > 2$. Multiplying (3.58) by ω_{1A}^1 and taking summation over A and then using (3.56) and (3.57), we get

$$4(s+t)t\lambda_n^2 - 4n(n-s-t+2)tH\lambda_n + n^2(n-t+2)(n-s-t+2)H^2 = 0. \tag{3.59}$$

Now, from (3.59), we find λ_n imaginary as discriminant $D = -16n^2H^2(n-s-t+2)(nst+2ts+2t^2) < 0$. Therefore, a contradiction, hence, $r > 2$ is not possible.

Now, for $r = 2$, (3.56) and (3.57) reduce to

$$-\frac{nH}{2}\{\frac{nH(n-s-t+2)}{2s} - \frac{t}{s}\lambda_n\} = 0, \tag{3.60}$$

and

$$-\frac{nH}{2}\lambda_n = 0. \tag{3.61}$$

Hence, From (3.60) and (3.61), we obtain that $H = 0$.

Case II: Let either $\lambda - \lambda_1 = 0$ or $\lambda_n - \lambda_1 = 0$ or $\lambda - \lambda_n = 0$. Then, from (3.45), we find that each eigen value λ, λ_1 and λ_n are proportional to H. So, from (3.46), we have

$$e_l(\lambda) = e_l(\lambda_1) = e_l(\lambda_n) = 0, \quad \text{for} \quad l = 2, 3, \ldots, n. \tag{3.62}$$

If $\lambda = \lambda_1$, then using (3.36), (3.62) and (3.5), we get

$$\omega_{CA}^C = \omega_{CC}^A = 0. \tag{3.63}$$

Using (3.63) and computing $g(R(e_C, e_1)e_C, e_2)$, we get that $H = 0$.

Now, if $\lambda_1 = \lambda_n$ or $\lambda = \lambda_n$, in both cases from (3.29), (3.62) and (3.5), we obtain $\omega_{BA}^B = \omega_{BB}^A = 0$. Evaluating $g(R(e_B, e_1)e_B, e_2)$, we find that $H = 0$.

Combining Case I and Case II, we have:

Proposition 3.1. *Let M_1^n be a biharmonic Lorentz hypersurface in the pseudo Euclidean space E_1^{n+1} having the non-diagonal shape operator given by (2.11). If grad$H$ is light like, then M_1^n is minimal.*

Now, we discuss the space like case of grad$H$.

Let grad$H$ be space like: In this case grad$H$ can be in the direction of e_A or e_B or e_C. In view of (3.42), (3.43) and (3.44), one of the multiplicities of eigen values must be one, otherwise, we get contradiction. Since $r \geq 2$, therefore either s or t must be one. Without loss of generality, we assume that $r \geq 2, s \geq 1, t = 1$ and grad$H$ is in the direction of e_n. We can write grad$H$= $e_n(H)e_n$. Now, we have $A = 3, 4, \ldots, r$, $B = r+1, r+2, \ldots, r+s = n-1$ and $C = n$. From (2.9) and (2.4), we get

$$\lambda_n = -\frac{nH}{2}, \quad \text{and} \quad \lambda_1 = \frac{3nH}{2(n-r-1)} - \frac{r\lambda}{n-r-1}. \tag{3.64}$$

Since grad$H$= $e_n(H)e_n$, therefore, from (3.64), we have

$$e_n(H) \neq 0, \quad e_a(H) = 0 \quad e_a(\lambda_n) = 0, \quad a = 1, 2, \ldots, n-1. \tag{3.65}$$

Using (3.4), (3.65) and the fact that $[e_a\ e_b](H) = 0 = \nabla_{e_a}e_b(H) - \nabla_{e_b}e_a(H)$, for $a \neq b$ and $a, b = 1, 2, \ldots, n-1$, we find

$$\omega_{ab}^n = \omega_{ba}^n. \tag{3.66}$$

Now, we consider the case of three distinct eigenvalues viz.

Case III: Let $\lambda - \lambda_1 \neq 0$, $\lambda_n - \lambda_1 \neq 0$ and $\lambda - \lambda_n \neq 0$.

From (3.6), (3.42), (3.64) and (3.65), we have

$$e_2(\lambda_1) = 0, \quad e_A(\lambda_1) = 0. \tag{3.67}$$

From (3.18), (3.24), (3.29), (3.36), (3.65), (3.67) and (3.5), we get

$$\omega_{B2}^B = \omega_{BB}^1 = \omega_{n2}^n = \omega_{nn}^1 = \omega_{BA}^B = \omega_{BB}^A = \omega_{nA}^n = \omega_{nn}^A = 0. \tag{3.68}$$

Using (3.7), (3.16), (3.17), (3.22), (3.26) and (3.5), we have

$$\omega_{2A}^1 = \omega_{22}^B = \omega_{2A}^B = \omega_{22}^n = \omega_{A2}^B = 0. \tag{3.69}$$

Using (3.20), (3.32), (3.66) and (3.5), we have

$$\omega_{2B}^n = \omega_{B2}^n = \omega_{2n}^B = \omega_{Bn}^1 = \omega_{A2}^n = \omega_{2A}^n = \omega_{2n}^A = 0. \tag{3.70}$$

Also, using (3.11), (3.31), (3.70), and (3.5), we obtain

$$\omega_{1B}^{n}=\omega_{B1}^{n}=\omega_{1n}^{B}=\omega_{Bn}^{2}=\omega_{AB}^{n}=\omega_{BA}^{n}=\omega_{An}^{B}=\omega_{Bn}^{A}=0. \tag{3.71}$$

From (3.28), (3.30), (3.34), (3.40), and (3.5), we get

$$\omega_{A\widetilde{A}}^{B}=\omega_{B\widetilde{B}}^{A}=\omega_{A\widetilde{A}}^{n}=\omega_{B\widetilde{B}}^{n}=0. \tag{3.72}$$

Using (3.14), (3.41), (3.65), (3.68) and (3.5), we have

$$\omega_{n1}^{n}=\omega_{nn}^{2}=\omega_{nB}^{n}=\omega_{nn}^{B}=0. \tag{3.73}$$

From (3.35), (3.38), (3.71), (3.70), and (3.5), we get

$$\omega_{nA}^{B}=\omega_{nB}^{1}=\omega_{nB}^{A}=\omega_{n2}^{B}=0. \tag{3.74}$$

From (3.10), (3.19), (3.37), (3.70), (3.71), (3.74), and (3.5), we get

$$\omega_{B1}^{\widetilde{B}}=\omega_{B\widetilde{B}}^{1}=\omega_{nB}^{2}=\omega_{n1}^{B}=0. \tag{3.75}$$

Now, we have the following:

Lemma 3.2. *Let M_1^n be a biharmonic Lorentz hypersurface in the pseudo Euclidean space E_1^{n+1}, having the non-diagonal shape operator given by (2.11). If grad$H$ is space like and in the direction of e_n, then*

$$\nabla_{e_1}e_2=\sum_{p\neq 1}\omega_{12}^{p}e_p,\ \ \nabla_{e_1}e_B=\sum_{p\neq B,n}\omega_{1B}^{p}e_p,\ \ \nabla_{e_1}e_n=\sum_{p\neq B,n}\omega_{1n}^{p}e_p,\ \ \nabla_{e_2}e_A=\sum_{p\neq 1,A,B,n}\omega_{2A}^{p}e_p,$$

$$\nabla_{e_2}e_B=\sum_{p\neq 1,A,B,n}\omega_{2B}^{p}e_p,\ \ \nabla_{e_2}e_n=\omega_{2n}^{2}e_2,\ \ \nabla_{e_A}e_B=\sum_{p\neq 1,\widetilde{A},B,n}\omega_{AB}^{p}e_p,\ \ \nabla_{e_A}e_{\widetilde{A}}=\sum_{p\neq\widetilde{A},B,n}\omega_{A\widetilde{A}}^{p}e_p,$$

$$\nabla_{e_A}e_n=\sum_{p\neq 1,\widetilde{A},B,n}\omega_{An}^{p}e_p,\ \ \nabla_{e_B}e_1=\sum_{p\neq 2,\widetilde{B},n}\omega_{B1}^{p}e_p,\ \ \nabla_{e_B}e_2=\sum_{p\neq 1,B,\widetilde{B},n}\omega_{B2}^{p}e_p,$$

$$\nabla_{e_B}e_A=\sum_{p\neq A,B,\widetilde{B},n}\omega_{BA}^{p}e_p,\ \ \nabla_{e_B}e_{\widetilde{B}}=\sum_{p\neq 1,A,\widetilde{B},n}\omega_{B\widetilde{B}}^{p}e_p,\ \ \nabla_{e_B}e_B=\sum_{p\neq 1,A,B}\omega_{BB}^{p}e_p,\ \ \nabla_{e_B}e_n=\omega_{Bn}^{B}e_B,$$

$$\nabla_{e_n}e_1=\sum_{p\neq 2,B,n}\omega_{n1}^{p}e_p,\ \ \nabla_{e_n}e_2=\sum_{p\neq 1,B,n}\omega_{n2}^{p}e_p,\ \ \nabla_{e_n}e_A=\sum_{p\neq A,B,n}\omega_{nA}^{p}e_p,\ \ \nabla_{e_n}e_B=\sum_{p=r+1}^{n-1}\omega_{nB}^{p}e_p,$$

$$\nabla_{e_n}e_n=0,\quad \nabla_{e_1}e_A=\sum_{p\neq A}\omega_{1A}^{p}e_p,\ \ \nabla_{e_2}e_2=\sum_{p\neq 1,B,n}\omega_{22}^{p}e_p,\ \ \nabla_{e_A}e_1=\sum_{p\neq 2}\omega_{A1}^{p}e_p,$$

$$\nabla_{e_A}e_2=\sum_{p\neq 1,B,n}\omega_{A2}^{p}e_p,\ \ \nabla_{e_2}e_1=\sum_{p\neq 2}\omega_{21}^{p}e_p.$$

Now, to find the Laplace operator, we need to construct an orthonormal basis $\{X_1,X_2,\dots,X_n\}$ from the pseudo-orthonormal basis $\{e_1,e_2,\dots,e_n\}$. Therefore, we take

$$X_1=\frac{e_1+e_2}{\sqrt{2}},\quad X_2=\frac{e_1-e_2}{\sqrt{2}},\quad X_i=e_i,\quad i=3,4,\dots,n. \tag{3.76}$$

Also, using (3.64), we obtain

$$\text{trace}S^2=\frac{(n-1)r}{n-r-1}\lambda^2+\frac{n^2(n-r+8)}{4(n-r-1)}H^2-\frac{3nr}{n-r-1}H\lambda. \tag{3.77}$$

Using (2.10) and (3.76) the Laplace operator for the pseudo-orthonormal basis $\{e_1, e_2, \ldots, e_n\}$, is given by

$$\triangle = e_1 e_2 + e_2 e_1 - \sum_{i=3}^{n} e_i e_i - \nabla_{e_1} e_2 - \nabla_{e_2} e_1 + \sum_{i=3}^{n} \nabla_{e_i} e_i (H). \tag{3.78}$$

Using (3.77), (3.78) and Lemma 3.3 in (2.8), we find

$$[-2\omega_{12}^n + \sum_{A=3}^{r} \omega_{AA}^n + \sum_{B=r+1}^{n-1} \omega_{BB}^n] e_n(H) - e_n e_n(H) + H[\frac{(n-1)r}{n-r-1}\lambda^2 + \frac{n^2(n-r+8)}{4(n-r-1)}H^2 - \frac{3nr}{n-r-1}H\lambda] = 0. \tag{3.79}$$

Now, from (3.12), (3.33), (3.39), (3.8), (3.27) and (3.5), we find

$$\omega_{AA}^n = -\omega_{12}^n, \quad \omega_{AA}^B = -\omega_{12}^B, \quad \omega_{AA}^n = \omega_{\tilde{A}\tilde{A}}^n, \quad \omega_{BB}^n = \omega_{\tilde{B}\tilde{B}}^n. \tag{3.80}$$

Therefore, using (3.80) in (3.79), we obtain

$$[-r\omega_{12}^n + (n-r-1)\omega_{BB}^n] e_n(H) - e_n e_n(H) + H[\frac{(n-1)r}{n-r-1}\lambda^2 + \frac{n^2(n-r+8)}{4(n-r-1)}H^2 - \frac{3nr}{n-r-1}H\lambda] = 0. \tag{3.81}$$

Now, we have:

Lemma 3.3. *Let M_1^n be a biharmonic Lorentz hypersurface in the pseudo Euclidean space E_1^{n+1}, having the non-diagonal shape operator given by (2.11). If $\text{grad}H$ is space like and in the direction of e_n. Then, $e_B(\lambda) = 0$ for $s \geq 1$.*

Proof. From (3.43) and (3.64), we get $e_B(\lambda) = 0$ for $s > 1$. Now, for $s = 1$, we have $B = n-1$ and $r = n-2$. Now, putting $r = n-2$ and $B = n-1$ in (3.81), we get

$$[(-n+2)\omega_{12}^n + \omega_{(n-1)(n-1)}^n] e_n(H) - e_n e_n(H) + H[(n-1)(n-2)\lambda^2 + \frac{5n^2}{2}H^2 - 3n(n-2)H\lambda] = 0. \tag{3.82}$$

Using (2.5), (3.5), (3.80) and Lemma 3.3, computing $g(R(e_{n-1}, e_1)e_2, e_n)$ and $g(R(e_A, e_{n-1})e_n, e_A)$, we find

$$e_{n-1}(\omega_{12}^n) + \omega_{12}^{n-1}(\omega_{(n-1)(n-1)}^n + \omega_{12}^n) - \sum_{A=3}^{r} \omega_{(n-1)A}^1 \omega_{1A}^n = 0, \tag{3.83}$$

and

$$e_{n-1}(\omega_{12}^n) + \omega_{12}^{n-1}(\omega_{(n-1)(n-1)}^n + \omega_{12}^n) + 2\omega_{(n-1)A}^1 \omega_{1A}^n = 0, \tag{3.84}$$

respectively.

Taking summation over A from 3 to r in (3.84), we find

$$(r-2)e_{n-1}(\omega_{12}^n) + (r-2)\omega_{12}^{n-1}(\omega_{(n-1)(n-1)}^n + \omega_{12}^n) + 2\sum_{A=3}^{r} \omega_{(n-1)A}^1 \omega_{1A}^n = 0. \tag{3.85}$$

Combining (3.83) and (3.85), we obtain

$$r e_{n-1}(\omega_{12}^n) + r\omega_{12}^{n-1}(\omega_{(n-1)(n-1)}^n + \omega_{12}^n) = 0, \tag{3.86}$$

or,

$$e_{n-1}(\omega_{12}^n) = -\omega_{12}^{n-1}(\omega_{(n-1)(n-1)}^n + \omega_{12}^n), \tag{3.87}$$

Using (3.8), (3.64), (3.87) and (3.5) for $r = n-2$, we find

$$e_{n-1}(\omega_{12}^n) = -\frac{e_{n-1}(\lambda)}{\frac{3nH}{2} - (n-1)\lambda}(\omega_{(n-1)(n-1)}^n + \omega_{12}^n). \tag{3.88}$$

Using (3.64), (3.65), (3.66) and $r = n-2$ in (3.12) and (3.39), we have

$$e_n(\lambda) = -(\frac{nH}{2} + \lambda)\omega_{12}^n \tag{3.89}$$

and

$$e_n(\frac{3nH}{2} - (n-2)\lambda) = (2nH - (n-2)\lambda)\omega_{(n-1)(n-1)}^n, \tag{3.90}$$

respectively.

Adding (3.89) and (3.90), we get

$$\frac{3n}{2}e_n(H) = -(n-2)(\frac{nH}{2} + \lambda)\omega_{12}^n + (2nH - (n-2)\lambda)\omega_{(n-1)(n-1)}^n. \tag{3.91}$$

Using (3.65) and Lemma 3.3, and the fact that $[e_a\ e_n](H) = 0 = \nabla_{e_a}e_n(H) - \nabla_{e_n}e_a(H)$, for $a = 1, 2, \ldots, n-1$, we obtain

$$e_a e_n(H) = 0. \tag{3.92}$$

Differentiating (3.91) with respect to e_{n-1} and using (3.88), (3.89) and (3.92), we find

$$e_{n-1}(\omega_{(n-1)(n-1)}^n) = \frac{2n(n-2)(H-\lambda)(\omega_{(n-1)(n-1)}^n + \omega_{12}^n)e_{n-1}(\lambda)}{(2nH - (n-2)\lambda)(3nH - 2(n-1)\lambda)}. \tag{3.93}$$

Taking derivative of (3.82) along e_{n-1} and using (3.88), (3.92) and (3.93), we get

$$(n-2)e_{n-1}(\lambda)\left[2e_n(H)(\omega_{(n-1)(n-1)}^n + \omega_{12}^n) + H(2(n-1)\lambda - 3nH)(2nH - (n-2)\lambda)\right] = 0.$$

If $e_{n-1}(\lambda) \neq 0$ in the above, then

$$2e_n(H)(\omega_{(n-1)(n-1)}^n + \omega_{12}^n) + H(2(n-1)\lambda - 3nH)(2nH - (n-2)\lambda) = 0. \tag{3.94}$$

Differentiating (3.94) along e_{n-1} and using (3.88) and (3.93), we obtain

$$4(n(n-4)H - (n-2)(n-1)\lambda)(\omega_{(n-1)(n-1)}^n + \omega_{12}^n)e_n(H) + H[n(7n-10)H - 4(n-1)(n-2)\lambda)(2nH - (n-2)\lambda)(3nH - 2(n-1)\lambda] = 0. \tag{3.95}$$

Eliminating $e_n(H)$ from (3.94) and (3.95), we get

$$\lambda = \tfrac{3nH}{2(n-1)} \Rightarrow \lambda_1 = \tfrac{3nH}{2(n-1)} = \lambda,$$

which is a contradiction of distinct principal curvatures, consequently $e_{n-1}(\lambda) = 0$. Whereby proof of Lemma is complete.

Next, we have:

Lemma 3.4. *Let M_1^n be a biharmonic Lorentz hypersurface in the pseudo Euclidean space E_1^{n+1}, having the non-diagonal shape operator given by (2.11). If grad$H$ is space like and in the direction of e_n. Then, we find*

$$e_n(\omega_{12}^n) + (\omega_{12}^n)^2 = \frac{nH}{2}\lambda, \tag{3.96}$$

$$\omega_{BB}^n\omega_{12}^n = \lambda(\frac{3nH}{2(n-r-1)} - \frac{r\lambda}{n-r-1}), \tag{3.97}$$

and

$$e_n(\omega_{BB}^n) - (\omega_{BB}^n)^2 = -\frac{nH}{2}(\frac{3nH}{2(n-r-1)} - \frac{r\lambda}{n-r-1}). \tag{3.98}$$

Proof. Using (3.8), (3.15), (3.16), (3.27), (3.5) and Lemma 3.4, we obtain

$$\omega_{12}^B = \omega_{21}^B = \omega_{1B}^1 = \omega_{2B}^2 = \omega_{AB}^A = \omega_{AA}^B = 0. \tag{3.99}$$

Also, evaluating $g((\nabla_{e_1}S)e_A, e_{\widetilde{A}}) = g((\nabla_{e_A}S)e_1, e_{\widetilde{A}})$ and $g((\nabla_{e_1}S)e_A, e_B) = g((\nabla_{e_A}S)e_1, e_B)$, using (2.6), (3.1) and (3.69), we get

$$\omega_{A2}^{\widetilde{A}} = \omega_{A\widetilde{A}}^{1} = 0 \quad \text{and} \quad \omega_{1A}^{B} = \omega_{A1}^{B}, \tag{3.100}$$

respectively.

Computing $g(R(e_n, e_1)e_n, e_2)$ and $g(R(e_n, e_A)e_n, e_A)$, using (2.5), (3.68), (3.99), (3.80), (3.5) and Lemma 3.3, we find

$$e_n(\omega_{12}^n) + (\omega_{12}^n)^2 - \sum_{A=3}^{r} \omega_{n2}^A \omega_{1A}^n = \frac{nH}{2}\lambda, \tag{3.101}$$

and

$$e_n(\omega_{12}^n) + (\omega_{12}^n)^2 + 2\omega_{n2}^A \omega_{1A}^n = \frac{nH}{2}\lambda, \tag{3.102}$$

respectively.

Now, taking summation over A from 3 to r in (3.102), we get

$$(r-2)e_n(\omega_{12}^n) + (r-2)(\omega_{12}^n)^2 + 2\sum_{A=3}^{r} \omega_{n2}^A \omega_{1A}^n = (r-2)\frac{nH}{2}\lambda. \tag{3.103}$$

Now, combining (3.101) and (3.103), we obtain (3.96).

Next, evaluating $g(R(e_1, e_B)e_B, e_2)$, $g(R(e_A, e_B)e_B, e_A)$ and $g(R(e_A, e_1)e_2, e_B)$, using (2.5), (3.68), (3.99), (3.100), (3.80), (3.5) and Lemma 3.3, we find

$$\omega_{BB}^n \omega_{12}^n - \sum_{A=3}^{r} \omega_{1B}^A \omega_{BA}^1 = \lambda(\frac{3nH}{2(n-r-1)} - \frac{r\lambda}{n-r-1}), \tag{3.104}$$

$$\omega_{BB}^n \omega_{12}^n + \omega_{BB}^2 \omega_{AA}^1 + 2\omega_{BA}^1 \omega_{1B}^A = \lambda(\frac{3nH}{2(n-r-1)} - \frac{r\lambda}{n-r-1}), \tag{3.105}$$

and

$$\omega_{AA}^1 \omega_{1B}^A = 0, \tag{3.106}$$

respectively.

From (3.106), we have either $\omega_{AA}^1 = 0$ or $\omega_{1B}^A = 0$. In both the cases, from (3.104) and (3.105), we get (3.97).

Similarly, evaluating $g(R(e_n, e_B)e_n, e_B)$, we obtain (3.98).

Now, we have:

Proposition 3.2. *Let M_1^n be a biharmonic Lorentz hypersurface in the pseudo Euclidean space E_1^{n+1} with three distinct eigen values and having the non-diagonal shape operator given by (2.11). If grad$H$ is space like, then M_1^n is not proper biharmonic.*

Proof. Using (3.64) and (3.5) in (3.12), we get

$$e_n(\lambda) = -(\frac{nH}{2} + \lambda)\omega_{12}^n. \tag{3.107}$$

Using (3.5), (3.64) and (3.107) in (3.39), we find

$$3ne_n(H) = [nH(n-r+2) - 2r\lambda]\omega_{BB}^n - 2r(\frac{nH}{2} + \lambda)\omega_{12}^n. \tag{3.108}$$

Now, multiplying (3.108) by ω_{12}^n and using (3.97), we have

$$(\omega_{12}^n)^2(\frac{nH}{2} + \lambda) = -3n\omega_{12}^n e_n(H) + \frac{\lambda}{n-r-1}[nH(n-r+2) - 2r\lambda](\frac{3nH}{2} - r\lambda). \tag{3.109}$$

Similarly, multiplying (3.108) by ω_{BB}^n and using (3.97), we obtain

$$(\omega_{BB}^n)^2(nH(n-r+2) - 2r\lambda) = 3n\omega_{BB}^n e_n(H) + \frac{2r\lambda}{n-r-1}(\frac{nH}{2} + \lambda)(\frac{3nH}{2} - r\lambda). \tag{3.110}$$

Differentiating (3.108) along e_n, and using (3.96), (3.97), (3.98) and (3.107), we find

$$3ne_ne_n(H) = e_n(H)[n(n-r+5)\omega_{BB}^n - n(r+6)\omega_{12}^n] - rnH\lambda(\frac{nH}{2}+\lambda) + \frac{3nH-2r\lambda}{4(n-r-1)}[-n^2(n-r+2)H^2 + 2n(r+2n+4)H\lambda]. \quad (3.111)$$

Eliminating $e_ne_n(H)$ from (3.81) and (3.111), we get

$$e_n(H)[(n-r-4)\omega_{BB}^n + (3-r)\omega_{12}^n] + \frac{3nH}{4(n-r-1)}[n(n-r+5)H^2 - (2n+8r+4)H\lambda + 4r\lambda^2] = 0. \quad (3.112)$$

Acting with e_n on (3.112) and putting the value of $e_ne_n(H)$ from (3.81) and using (3.96), (3.97), (3.98) and (3.107), we find

$$[(n-r-4)\omega_{BB}^n + (3-r)\omega_{12}^n][H\{\frac{(n-1)r}{n-r-1}\lambda^2 - \frac{3nr}{n-r-1}H\lambda + \frac{n^2(n-r+8)}{4(n-r-1)}\}] + [(n-r)(n-r-4)(\omega_{BB}^n)^2 - (3-r)(r+1)(\omega_{12}^n)^2 + \frac{[(n-r-1)(3-r)-r(n-r-4)]}{2(n-r-1)} (3nH\lambda - 2r\lambda^2)]e_n(H) + \frac{ne_n(H)}{4(n-r-1)}[3n(2n-2r+19)H^2 + (2r(n-r-4)- 2(6n+25r+9))H\lambda + 12r\lambda^2] + \frac{3n}{4(n-r-1)}[(n+4r+2)H^2 - 4rH\lambda](nH+2\lambda)\omega_{12}^n = 0. \quad (3.113)$$

Now, multiplying (3.112) by ω_{12}^n and using (3.97), we have

$$(3-r)e_n(H)(\omega_{12}^n)^2 = -\frac{n-r-4}{2(n-r-1)}(3nH\lambda - 2r\lambda^2)e_n(H) - \frac{3nH}{4(n-r-1)}[n(n-r+5)H^2 - (2n+8r+4)H\lambda + 4r\lambda^2]\omega_{12}^n. \quad (3.114)$$

Similarly, multiplying (3.112) by ω_{BB}^n and using (3.97), we obtain

$$(n-r-4)e_n(H)(\omega_{BB}^n)^2 = -\frac{3-r}{2(n-r-1)}(3nH\lambda - 2r\lambda^2)e_n(H) - \frac{3nH}{4(n-r-1)}[n(n-r+5)H^2 - (2n+8r+4)H\lambda + 4r\lambda^2]\omega_{BB}^n. \quad (3.115)$$

Using (3.114) and (3.115) in (3.113), we get

$$\omega_{12}^n E + \omega_{BB}^n F + e_n(H)G = 0, \quad (3.116)$$

where

$$E = H[(9n+13r+2nr-2r^2+45)n^2H^2 + 4r(2nr+r-3)\lambda^2 - 6nr(3n+2r+8)H\lambda],$$
$$F = H[-((n-r)(2n-2r+11)+32)n^2H^2 - 4r((n-r)(2n+1)-4(n-1))\lambda^2 + 6n((n-r)(n+2r+8)+8r)H\lambda],$$
$$G = 4r(2n+7)\lambda^2 + 3n^2(2n-2r+19)H^2 + 2n((n-r)(r-6)+(3n-35r-30))H\lambda.$$

Eliminating $e_n(H)$ from (3.116) and (3.108), we obtain

$$\omega_{12}^n f_1(H,\lambda) + \omega_{BB}^n f_2(H,\lambda) = 0, \quad (3.117)$$

where $f_1(H,\lambda) = E - \frac{r(nH+2\lambda)}{3n}G$ and $f_2(H,\lambda) = F + \frac{(n(n-r+2)H-2r\lambda)}{3n}G$ are the homogeneous functions of degree 3 in terms of H and λ.

Multiplying (3.117) by ω_{12}^n and ω_{BB}^n and using (3.97), we obtain

$$(\omega_{12}^n)^2 f_1(H,\lambda) = -\frac{\lambda}{2(n-r-1)}(3nH - 2r\lambda)f_2(H,\lambda), \quad (3.118)$$

and

$$(\omega_{BB}^n)^2 f_2(H,\lambda) = -\frac{\lambda}{2(n-r-1)}(3nH - 2r\lambda)f_1(H,\lambda), \tag{3.119}$$

respectively.

Again, eliminating $e_n(H)$ from (3.112) and (3.108), we get

$$P_1(\omega_{BB}^n)^2 - P_2(\omega_{12}^n)^2 + P_3 = 0, \tag{3.120}$$

where

$$\begin{aligned}
P_1 &= 4(n-r-1)(n-r-4)(n(n-r+2)H - 2r\lambda),\\
P_2 &= 4r(n-r-1)(3-r)(nH + 2\lambda),\\
P_3 &= 8r^2(n-2r-1)\lambda^3 + 9n^3(n-r+5)H^3 - 6n^2r(2n-2r+13)H^2\lambda\\
&\quad + 4nr\{3(n+3r-1) + 2r(n-r-1)\}H\lambda^2.
\end{aligned}$$

Now, eliminating ω_{12}^n and ω_{BB}^n from (3.120) by using (3.118) and (3.119), we obtain

$$\lambda(3nH - 2r\lambda)[(f_2(H,\lambda))^2 P_2 - (f_1(H,\lambda))^2 P_1] + 2(n-r-1)f_1(H,\lambda)f_2(H,\lambda)P_3 = 0, \tag{3.121}$$

which is a homogeneous equation of degree 9 in terms of H and λ. Here, we point out that $\lambda \neq 0$. In fact, if $\lambda = 0$ then (3.121) gives $H = 0$, which is contradiction to our assumption. We put $Y = \frac{H}{\lambda}$, then (3.121) will reduce to an algebraic equation in Y

$$(3nY - 2r)[P_4 - P_5] + 2(n-r-1)P_6 = 0, \tag{3.122}$$

where

$$\begin{aligned}
P_4 &= 4r(n-r-1)(3-r)(nY+2)(g_2(Y))^2,\\
P_5 &= 4(n-r-1)(n-r-4)(n(n-r+2)Y - 2r)(g_1(Y))^2,\\
P_6 &= [8r^2(n-2r-1) + 9n^3(n-r+5)Y^3 - 6n^2r(2n-2r+13)Y^2\\
&\quad + 4nr\{3(n+3r-1) + 2r(n-r-1)\}Y]g_1(Y)g_2(Y),\\
g_1(Y) &= Y[(9n + 13r + 2nr - 2r^2 + 45)n^2Y^2 + 4r(2nr + r - 3) - 6nr(3n + 2r + 8)Y]\\
&\quad - \frac{r(nY+2)}{3n}[4r(2n+7) + 3n^2(2n-2r+19)Y^2 + 2n((n-r)(r-6)\\
&\quad + (3n - 35r - 30))Y],\\
g_2(Y) &= Y[-((n-r)(2n-2r+11) + 32)n^2Y^2 - 4r((n-r)(2n+1) - 4(n-1))\\
&\quad + 6n((n-r)(n+2r+8) + 8r)Y] + \frac{(n(n-r+2)Y - 2r)}{3n}[4r(2n+7)\\
&\quad + 3n^2(2n-2r+19)Y^2 + 2n((n-r)(r-6) + (3n - 35r - 30))Y].
\end{aligned}$$

and without having solve to (3.122) explicitly, even in the case of the existence of a real solution, H will be proportional to λ with a numerical factor ν, where ν be the root of the equation (3.122). Hence, we can assume that $H = \nu\lambda$ and substituting it in (3.107) and (3.108), and using (3.96), (3.97) and (3.98), we obtain

$$-\lambda e_n e_n(\lambda) + \frac{e_n^2(\lambda)(n\nu+4)}{n\nu+2} = \frac{n\nu(n\nu+2)\lambda^4}{4}, \tag{3.123}$$

$$e_n^2(\lambda) = -\frac{(n\nu+2)(n(n-r+2)\nu - 2r)\lambda^4}{4(n-r-1)}, \tag{3.124}$$

$$\lambda e_n e_n(\lambda) - e_n^2(\lambda)\left(1 + \frac{3\nu n - 2r}{n(n-r+2)\nu - 2r}\right) = -\frac{n\nu(n(n-r+2)\nu - 2r)\lambda^4}{4(n-r-1)}. \tag{3.125}$$

Adding (3.123) and (3.125), we find

$$e_n^2(\lambda) = \frac{(n\nu+2)(n(n-r+2)\nu - 2r)\lambda^4}{4(n-r-1)}. \tag{3.126}$$

Using (3.124) and (3.126), we get $e_n(\lambda) = 0$. Since $H = \nu\lambda$, therefore we obtain $e_n(H) = 0$, a contradiction to (3.65). Which completes the proof of Proposition 3.6.

Now, we consider the case of two distinct eigenvalues.

Case IV: Let either of $\lambda - \lambda_1 = 0$ or $\lambda_n - \lambda_1 = 0$ or $\lambda - \lambda_n = 0$. Then, from (3.64), we can say that each eigen value λ, λ_1 and λ_n is the multiple of H. From (3.65), we have

$$e_a(\lambda) = e_a(\lambda_1) = e_a(\lambda_n) = 0, \quad \text{for} \quad a = 1, 2, ..., n-1. \tag{3.127}$$

If $\lambda - \lambda_n = 0$ or $\lambda_n - \lambda_1 = 0$, then from (3.33) or (3.39), we get $e_n(H) = 0$ which is a contradiction to (3.65). Now, if $\lambda - \lambda_1 = 0$, then $r = n - 1$. From (3.64), we have

$$\lambda = \lambda_1 = \frac{3nH}{2(n-1)}. \tag{3.128}$$

Putting $r = n - 1$ in (3.81) and using (3.128), we get

$$-(n-1)\omega_{12}^n e_n(H) - e_n e_n(H) + \frac{n^2(n+8)}{4(n-1)} H^3 = 0. \tag{3.129}$$

Using (3.128) in (3.96), we find

$$e_n(\omega_{12}^n) + (\omega_{12}^n)^2 = \frac{3n^2 H^2}{4(n-1)}. \tag{3.130}$$

Using (3.5), (3.64) and (3.128) in (3.12), we have

$$e_n(H) = -\frac{n+2}{3} H \omega_{12}^n. \tag{3.131}$$

Differentiating (3.131) along e_n and using (3.125), we get

$$e_n e_n(H) = \frac{(n+2)(n+5)}{9} H (\omega_{12}^n)^2 - \frac{n^2(n+2)}{4(n-1)} H^3. \tag{3.132}$$

Eliminating $e_n e_n(H)$ from (3.129) and (3.132), we obtain

$$\frac{2(n+2)(n-4)}{9} (\omega_{12}^n)^2 + \frac{n^2(n+5)}{2(n-1)} H^2 = 0. \tag{3.133}$$

Differentiating, again (3.133) along e_n and using (3.130) and (3.131), we get

$$\frac{4(n-4)}{9} (\omega_{12}^n)^2 + \frac{3n^2}{(n-1)} H^2 = 0. \tag{3.134}$$

Therefore, from (3.133) and (3.134), we can conclude that H must be zero.

Combining Proposition 3.6 and Case IV, we have

Proposition 3.3. *Let M_1^n be a biharmonic Lorentz hypersurface in the pseudo Euclidean space E_1^{n+1}, having the non-diagonal shape operator given by (2.11). If grad$H$ is space like, then M_1^n is not proper biharmonic.*

Now, using Propositions 3.2 and 3.7, we have following:

Theorem 3.1. *Let M_1^n be a biharmonic Lorentz hypersurface in the pseudo Euclidean space E_1^{n+1}, having non-diagonal shape operator given by (2.11) with at most three distinct principal curvatures. Then M_1^n is not proper biharmonic.*

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