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Classification of Rectifying Space-Like Submanifolds in Pseudo-Euclidean Spaces

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ABSTRACT

The notions of rectifying subspaces and of rectifying submanifolds were introduced in [B.-Y. Chen, Int. Electron. J. Geom 9 (2016), no. 2, 1–8]. More precisely, a submanifold in a Euclidean m-space $\mathbb{E}^m$ is called a rectifying submanifold if its position vector field always lies in its rectifying subspace. Several fundamental properties and classification of rectifying submanifolds in Euclidean space were obtained in [B.-Y. Chen, op. cit.].
In this present article, we extend the results in [B.-Y. Chen, op. cit.] to rectifying space-like submanifolds in a pseudo-Euclidean space with arbitrary codimension. In particular, we completely classify all rectifying space-like submanifolds in an arbitrary pseudo-Euclidean space with codimension greater than one.

Keywords: *Rectifying submanifold; rectifying subspace; pseudo-Euclidean space; concurrent vector field; space-like submanifold; position vector field.*
AMS Subject Classification (2010): *Primary: 53C40; Secondary: 53C42.*

1. Introduction

Let $\mathbb{E}^3$ denote the Euclidean 3-space with its inner product $\langle\ ,\ \rangle$. Consider a unit-speed space curve $x : I \to \mathbb{E}^3$, where $I = (\alpha, \beta)$ is a real interval. Let $\mathbf{x}$ denote the position vector field of x and $\mathbf{x}'$ be denoted by $\mathbf{t}$.

It is possible, in general, that $\mathbf{t}'(s) = 0$ for some s; however, we assume that this never happens. Then we can introduce a unique vector field $\mathbf{n}$ and positive function κ so that $\mathbf{t}' = \kappa\mathbf{n}$. We call $\mathbf{t}'$ the *curvature vector field*, $\mathbf{n}$ the *principal normal vector field*, and κ the *curvature* of the curve. Since $\mathbf{t}$ is of constant length, $\mathbf{n}$ is orthogonal to $\mathbf{t}$. The *binormal vector field* is defined by $\mathbf{b} = \mathbf{t} \times \mathbf{n}$, which is a unit vector field orthogonal to both $\mathbf{t}$ and $\mathbf{n}$. One defines the *torsion* τ by the equation $\mathbf{b}' = -\tau\mathbf{n}$.

The famous Frenet-Serret equations are given by

$$\begin{cases} \mathbf{t}' = & \kappa\mathbf{n} \\ \mathbf{n}' = -\kappa\mathbf{t} & \quad + \tau\mathbf{b} \\ \mathbf{b}' = & -\tau\mathbf{n}. \end{cases} \tag{1.1}$$

At each point of the curve, the planes spanned by $\{\mathbf{t}, \mathbf{n}\}$, $\{\mathbf{t}, \mathbf{b}\}$, and $\{\mathbf{n}, \mathbf{b}\}$ are known as the *osculating plane*, the *rectifying plane*, and the *normal plane*, respectively.

From elementary differential geometry it is well known that a curve in $\mathbb{E}^3$ lies in a plane if its position vector lies in its osculating plane at each point, and it lies on a sphere if its position vector lies in its normal plane at each point. A curve in the Euclidean 3-space is called a rectifying curve if if its position vector field always lies in its rectifying plane (cf. [3]). Rectifying curves have many interesting properties. Such curves have been studied by many authors, see for instance, [1, 3, 10, 9, 13, 14, 15] among many others.

In [6], the first author introduced the notion of rectifying subspaces for Euclidean submanifolds. As a natural extension of rectifying curves, the first author defined the notion of rectifying submanifolds as Euclidean submanifolds whose position vector field always lie in its rectifying subspace [6]. Many fundamental properties of rectifying submanifolds are obtained in [6, 7]. In particular, the first author proved that a Euclidean

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submanifold is rectifying if and only if the tangential component of its position vector field is a concurrent vector field. Furthermore, he completely determined rectifying submanifolds in a Euclidean space with arbitrary codimension.

In this article we extend the results of [6] to rectifying space-like submanifolds in a pseudo-Euclidean space with arbitrary codimension as a supplement to [6]. In particular, we completely classify all rectifying space-like submanifolds in an arbitrary pseudo-Euclidean space.

2. Preliminaries

For general references on submanifolds in pseudo-Riemannian manifolds, we refer to [5, 8, 16].

Let $\mathbb{E}^m_i$ denote the pseudo-Euclidean m-space equipped with the canonical pseudo-Euclidean metric g_0 of index i given by

$$g_0 = -\sum_{r=1}^{i} du_r^2 + \sum_{t=i+1}^{m} du_t^2, \tag{2.1}$$

where $(u_1, \dots, u_m)$ is a rectangular coordinate system of $\mathbb{E}^m_i$.

Let $x : M \to \mathbb{E}^m_i$ be an isometric immersion of a pseudo-Riemannian n-manifold M into $\mathbb{E}^m_i$. For a point $p \in M$, we denote by T_pM and $T_p^{\perp}M$ the tangent and the normal spaces at p. There is a natural orthogonal decomposition:

$$T_p\mathbb{E}^m_i = T_pM \oplus T_p^{\perp}M. \tag{2.2}$$

Denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and $\mathbb{E}^m_i$, respectively. The formulas of Gauss and Weingarten are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.3}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.4}$$

for vector fields X, Y tangent to M and ξ normal to M, where h is the second fundamental form, D the normal connection, and A the shape operator of M.

For a given point $p \in M$, the *first normal space*, of M in $\mathbb{E}^m_i$, denoted by $\operatorname{Im} h_p$, is the subspace defined by

$$\operatorname{Im} h_p = \operatorname{Span}\{h(X, Y) : X, Y \in T_pM\}. \tag{2.5}$$

For each normal vector ξ at p, the shape operator A_ξ is an endomorphism of T_pM. The second fundamental form h and the shape operator A are related by

$$\langle A_\xi X, Y \rangle = \langle h(X, Y), \xi \rangle, \tag{2.6}$$

where $\langle\ ,\ \rangle$ denotes the scalar product on M as well as on the ambient space.

The *equation of Gauss* of M in $\mathbb{E}^m_i$ is given by

$$R(X, Y; Z, W) = \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle \tag{2.7}$$

for X, Y, Z, W tangent to M, where R denotes the curvature tensors of M.

The covariant derivative $\bar{\nabla} h$ of h with respect to the connection on $TM \oplus T^{\perp}M$ is defined by

$$(\bar{\nabla}_X h)(Y, Z) = D_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.8}$$

The *equation of Codazzi* is

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z). \tag{2.9}$$

It follows from the definition of a rectifying curve $x : I \to \mathbb{E}^3$ that the position vector field $\mathbf{x}$ of x satisfies

$$\mathbf{x}(s) = \lambda(s)\mathbf{t}(s) + \mu(s)\mathbf{b}(s) \tag{2.10}$$

for some functions λ and μ.

For a curve $x : I \to \mathbb{E}^3$ with $\kappa(s_0) \neq 0$ at $s_0 \in I$, the first normal space at s_0 is the line spanned by the principal normal vector $\mathbf{n}(s_0)$. Hence, the rectifying plane at s_0 is nothing but the plane orthogonal to the first normal space at s_0. Therefore, for a submanifold M of $\mathbb{E}^m_i$ and a point $p \in M$, we call the subspace of $T_p\mathbb{E}^m_i$, orthogonal complement to the first normal space $\operatorname{Im} h_p$, the *rectifying space of M* at p (see [6]).

We make the following definition as in [6].

Definition 2.1. A pseudo-Riemannian submanifold M of a pseudo-Euclidean space $\mathbb{E}^m_i$ is called a *rectifying submanifold* if the position vector field $\mathbf{x}$ of M always lies in its rectifying space. In other words, M is a rectifying submanifold if and only if

$$\langle \mathbf{x}(p), \operatorname{Im} h_p \rangle = 0 \tag{2.11}$$

holds at every $p \in M$.

3. Lemmas

A tangent vector v of a pseudo-Riemannian manifold $\tilde{M}^m_i$ is called *space-like* (respectively, *time-like*) if $v = 0$ or $\langle v, v\rangle > 0$ (respectively, $\langle v, v\rangle < 0$). A vector v is called *light-like* or *null* if $v \neq 0$ and $\langle v, v\rangle = 0$.

The *light cone $\mathcal{LC}$* of $\mathbb{E}^m_i$ is defined by

$$\mathcal{LC} = \{v \in \mathbb{E}^m_i : \langle v, v\rangle = 0\}. \tag{3.1}$$

Let r be a positive number. We put

$$S^k_i(r^2) = \left\{\mathbf{x} \in \mathbb{E}^{k+1}_i : \langle \mathbf{x}, \mathbf{x}\rangle = r^2\right\}, \quad i > 0, \tag{3.2}$$

$$H^k_i(-r^2) = \left\{\mathbf{x} \in \mathbb{E}^{k+1}_{i+1} : \langle \mathbf{x}, \mathbf{x}\rangle = -r^2\right\}, \quad i > 0, \tag{3.3}$$

$$H^k(c) = \left\{\mathbf{x} \in \mathbb{E}^{k+1}_1 : \langle \mathbf{x}, \mathbf{x}\rangle = -r^2 \text{ and } x_1 > 0\right\}, \tag{3.4}$$

$S^k_i(r^2)$ (respectively, $H^k_i(-r^2)$) is a pseudo-Riemannian manifolds of curvature $1/r^2$ (respectively, $-1/r^2$) with index i. The $S^k_i(r^2)$ (respectively, $H^k_i(-r^2)$) is known as a *pseudo-sphere* (respectively, *pseudo-hyperbolic space*).

The pseudo-Riemannian manifolds $\mathbb{E}^k_i$, $S^k_i(r^2)$, $H^k_i(-r^2)$ are the standard models of the *indefinite real space forms*. In particular, $\mathbb{E}^k_1$, $S^k_1(c)$, $H^k_1(c)$ are the standard models of *Lorentzian space forms*

A submanifold M of $\mathbb{E}^m_i$ is called *space-like* if each tangent vector of M is space-like.

By a *cone in $\mathbb{E}^m_i$ with vertex at the origin $o \in \mathbb{E}^m_i$* we mean a ruled submanifold generated by a family of half lines initiated at o. A submanifold of $\mathbb{E}^m_i$ is called a *conic submanifold* with vertex at o if it is an open portion of a cone with vertex at o.

For a space-like submanifold M of $\mathbb{E}^m_i$, there exists a natural orthogonal decomposition of the position vector field $\mathbf{x}$ at each point; namely,

$$\mathbf{x} = \mathbf{x}^T + \mathbf{x}^N, \tag{3.5}$$

where $\mathbf{x}^T$ and $\mathbf{x}^N$ denote the tangential and normal components of $\mathbf{x}$, respectively.

We put

$$|\mathbf{x}^T|^2 = \left\langle \mathbf{x}^T, \mathbf{x}^T\right\rangle, \quad |\mathbf{x}^N|^2 = \left\langle \mathbf{x}^N, \mathbf{x}^N\right\rangle.$$

Lemma 3.1. *Let M be a pseudo-Riemannian submanifold of the pseudo-Euclidean space $\mathbb{E}^m_i$. If the position vector field $\mathbf{x}$ of M in $\mathbb{E}^m_i$ is either space-like or time-like, then $\mathbf{x} = \mathbf{x}^T$ holds identically if and only if M is a conic submanifold with the vertex at the origin.*

Proof. Let M be a pseudo-Riemannian submanifold of $\mathbb{E}^m_i$. Assume that the position vector field $\mathbf{x}$ of M in $\mathbb{E}^m_i$ is either space-like or time-like. If $\mathbf{x} = \mathbf{x}^T$ holds identically, then $e_1 = \mathbf{x}/|\mathbf{x}|$ is a unit vector field.

Put $\mathbf{x} = \rho e_1$. Then we get

$$\tilde{\nabla}_{e_1}\mathbf{x} = e_1, \quad \tilde{\nabla}_{e_1}\mathbf{x} = (e_1\rho)e_1 + \rho\tilde{\nabla}_{e_1}e_1. \tag{3.6}$$

Since $\tilde{\nabla}_{e_1}e_1$ is perpendicular to e_1, we find from (3.6) that $\tilde{\nabla}_{e_1}e_1 = 0$. Therefore the integral curves of e_1 are some open portions of generating lines in $\mathbb{E}^m$. Moreover, because $\mathbf{x} = \mathbf{x}^T$, the generating lines given by the integral curves of e_1 pass through the origin. Consequently, M is a conic submanifold with the vertex at the origin.

The converse is clear. ∎

We recall the following definition of concurrent vector fields.

Definition 3.1. A non-trivial vector field C on a Riemannian (or more generally, on a pseudo-Riemannian) manifold M is called a *concurrent vector field* if it satisfies

$$\nabla_X C = X \tag{3.7}$$

for any vector X tangent to M, where ∇ is the Levi-Civita connection of M.

Remark 3.1. Since the position vector field of the pseudo-Euclidean space $\mathbb{E}^m_i$ is a concurrent vector field, it follows that the position vector field $\mathbf{x}$ of any pseudo-Riemannian submanifold M in $\mathbb{E}^m_i$ satisfies

$$\tilde{\nabla}_Z \mathbf{x} = Z \tag{3.8}$$

for any $Z \in TM$, where $\tilde{\nabla}$ is the Levi-Civita connection of $\mathbb{E}^m_i$.

Lemma 3.2. *Let M be a pseudo-Riemannian submanifold of $\mathbb{E}^m_i$. If the position vector field $\mathbf{x}$ is either space-like or time-like, then the position vector field $\mathbf{x}$ of M satisfies $\mathbf{x} = \mathbf{x}^N$ identically if and only if M lies in one of the following hypersurfaces of $\mathbb{E}^m_i$:*

(1) *a pseudo-sphere $S^{m-1}_i(c^2)$; or*
(2) *a pseudo-hyperbolic space $H^{m-1}_{i-1}(-c^2)$ whenever $i > 1$; or*
(3) *a hyperbolic space $H^{m-1}(-c^2)$ whenever $i = 1$,*

where c is a positive number.

Proof. Let $x : M \to \mathbb{E}^m_i$ be an isometric immersion of a pseudo-Riemannian n-manifold into $\mathbb{E}^m_i$ with space-like or time-like position vector field. If $\mathbf{x} = \mathbf{x}^N$ holds identically, then we get from (3.8) that

$$Z\langle \mathbf{x}, \mathbf{x}\rangle = 2\,\langle \tilde{\nabla}_Z \mathbf{x}, \mathbf{x}\rangle = 2\left\langle Z, \mathbf{x}^N\right\rangle = 0$$

for any $Z \in TM$. Thus M lies in one of the three hypersurfaces of $\mathbb{E}^m_i$.

The converse is easy to verify. □

In views of Lemma 3.1 and Lemma 3.2 we make the following.

Definition 3.2. A rectifying submanifold M of $\mathbb{E}^m_i$ is called *proper* if its position vector field $\mathbf{x}$ satisfies $\mathbf{x} \neq \mathbf{x}^T$ and $\mathbf{x} \neq \mathbf{x}^N$ at every point on M.

In this article, we are only interested on proper rectifying submanifolds of $\mathbb{E}^m_i$ in views of Lemma 3.1 and Lemma 3.2.

For the proof of our main theorem we also need the following lemma.

Lemma 3.3. *Let M be a pseudo-Riemannian submanifold of $\mathbb{E}^m_i$. If M is proper rectifying, then $\left\langle \mathbf{x}^N, \mathbf{x}^N\right\rangle$ is constant on M.*

Proof. Let $x : M \to \mathbb{E}^m_i$ be an isometric immersion of a Riemannian n-manifold into $\mathbb{E}^m_i$. Consider the orthogonal decomposition

$$\mathbf{x} = \mathbf{x}^T + \mathbf{x}^N \tag{3.9}$$

of the position vector field $\mathbf{x}$ of M in $\mathbb{E}^m_i$. It follows from (3.9) and the formula of Gauss and the formula of Weingarten that

$$Z = \tilde{\nabla}_Z \mathbf{x} = \nabla_Z \mathbf{x}^T + h(Z, \mathbf{x}^T) - A_{\mathbf{x}^N} Z + D_Z \mathbf{x}^N \tag{3.10}$$

for any $Z \in TM$. By comparing the normal components in (3.10), we find

$$D_Z \mathbf{x}^N = -h(Z, \mathbf{x}^T). \tag{3.11}$$

Therefore we obtain

$$Z\left\langle \mathbf{x}^N, \mathbf{x}^N\right\rangle = 2\left\langle D_Z \mathbf{x}^N, \mathbf{x}^N\right\rangle = -\left\langle h(Z, \mathbf{x}^T), \mathbf{x}\right\rangle = 0, \tag{3.12}$$

where we have used (2.11) in Definition 2.1. Since (3.12) holds identically for any $Z \in TM$, we conclude that $\left\langle \mathbf{x}^N, \mathbf{x}^N\right\rangle$ is constant on M. □

Remark 3.2. A submanifold M of $\mathbb{E}^m_i$ is called a *T-submanifold* (respectively, *N-submanifold*) if its position vector field $\mathbf{x}$ satisfies $\left\langle \mathbf{x}^T, \mathbf{x}^T\right\rangle = constant$ (respectively, $\left\langle \mathbf{x}^N, \mathbf{x}^N\right\rangle = constant$) (cf. [2, 4]). Obviously, Lemma 3.3 implies that every proper rectifying pseudo-Riemannian submanifold of $\mathbb{E}^m_i$ is an N-submanifold.

4. Characterization of rectifying submanifolds in $\mathbb{E}^m_i$

The following result provides a very simple characterization of rectifying submanifolds.

Theorem 4.1. *If the position vector field $\mathbf{x}$ of a pseudo-Riemannian submanifold M in $\mathbb{E}^m_i$ satisfies $\mathbf{x}^N \neq 0$, then M is a proper rectifying submanifold if and only if $\mathbf{x}^T$ is a concurrent vector field on M.*

Proof. Let M be a space-like submanifold of $\mathbb{E}^m_i$. Then (3.10) holds. After comparing the tangential components in (3.10), we obtain

$$A_{\mathbf{x}^N} Z = \nabla_Z \mathbf{x}^T - Z. \tag{4.1}$$

Assume that M is a proper rectifying submanifold of $\mathbb{E}^m_i$. Then we have $\mathbf{x}^T \neq 0$ and $\mathbf{x}^N \neq 0$. Moreover, it follows from the Definition 2.1 that

$$\langle A_{\mathbf{x}^N} X, Y\rangle = \langle \mathbf{x}, h(X,Y)\rangle = 0 \tag{4.2}$$

for $X, Y \in TM$. Since M is space-like, we find from (4.1) that $A_{\mathbf{x}^N} = 0$. Therefore (3.8) yields

$$\nabla_Z \mathbf{x}^T = Z, \tag{4.3}$$

for any $Z \in TM$. Consequently, $\mathbf{x}^T$ is a concurrent vector field on M.

Conversely, if $\mathbf{x}^T$ is a concurrent vector field on M, then (3.7) and (4.1) give $A_{\mathbf{x}^N} = 0$. Therefore we obtain (4.3). Consequently, M is a proper rectifying submanifold due to $\mathbf{x}^N \neq 0$ by assumption. □

The next result shows that every proper rectifying space-like submanifold is a warped product.

Theorem 4.2. *Let M be a proper rectifying space-like submanifold M of $\mathbb{E}^m_i$. Then M is a warped product manifold $I \times_s F$ with warping metric*

$$g = ds^2 + s^2 g_F, \tag{4.4}$$

such that $\mathbf{x}^T = s\partial/\partial s$ and g_F is the metric tensor of a Riemannian manifold F.

Proof. Let M be a proper rectifying space-like submanifold of $\mathbb{E}^m_i$. Then we have $\mathbf{x}^T \neq 0$ and $\mathbf{x}^N \neq 0$. Thus we may put

$$\mathbf{x}^T = \rho e_1, \quad \rho = |\mathbf{x}^T| > 0, \tag{4.5}$$

where e_1 is a space-like unit vector field. We may extend e_1 to a local orthonormal frame $e_1, e_2, \ldots, e_n$ on M.

Obviously, it follows from (4.5) that $\rho = \langle \mathbf{x}, e_1\rangle$. Thus, by taking the derivative of ρ with respect to e_j for $j = 1, \ldots, n$ and using (2.3) and (3.8), we find

$$e_j \rho = \delta_{1j} + \langle \mathbf{x}, h(e_1, e_j)\rangle , \tag{4.6}$$

where $\delta_{ij} = 1$ or 0 depending on $i = j$ or $i \neq j$. Combining (2.11) and (4.6) gives

$$e_1\rho = 1, \quad e_2\rho = \cdots = e_n\rho = 0.$$

Therefore we get $\rho = \rho(s)$ and $\rho'(s) = 1$, which imply $\rho(s) = s + b$ for some real number b. Hence, after applying a suitable translation on s if necessary, we have $\rho = s$. Therefore, we obtain

$$\mathbf{x}^T = s e_1 = s\frac{\partial}{\partial s}. \tag{4.7}$$

Since M is a proper rectifying space-like submanifold, Theorem 4.1 implies that $\mathbf{x}^T = s e_1$ is a concurrent vector field. Thus we find from (4.3) that

$$e_1 = \nabla_{e_1} \mathbf{x}^T = \nabla_{e_1} s e_1 = e_1 + s\nabla_{e_1} e_1, \tag{4.8}$$

which implies $\nabla_{e_1} e_1 = 0$. Therefore the integral curves of e_1 are geodesics of M. Consequently, the distribution $\mathcal{D}^\perp$ spanned by e_1 is a totally geodesic foliation.

From (4.3) we also find
$$e_i = \nabla_{e_i}\mathbf{x}^T = s\nabla_{e_i}e_1, \quad i = 2, \ldots, n, \tag{4.9}$$
which gives
$$\omega_1^j(e_i) = \frac{\delta_{ij}}{s}, \quad i, j = 2, \ldots, n. \tag{4.10}$$
We conclude from (4.10) that the distribution $\mathcal{D}$ is integrable whose leaves are totally umbilical hypersurfaces of M. Moreover, it follows from (4.10) that the mean curvature of leaves of $\mathcal{D}$ are given by s^{-1}. Since the leaves of $\mathcal{D}$ are hypersurfaces, it follows that the mean curvature vector field of the leaves of $\mathcal{D}_2$ is parallel in the normal bundle in M. Therefore the distribution $\mathcal{D}$ is a spherical foliation. Consequently, by applying a result of [12] (or Theorem 4.4 of [5, page 90]) we conclude that M is locally a warped product $I \times_s F$, where F is a Riemannian $(n-1)$-manifold. Therefore the metric tensor g of M takes the form (4.4). □

5. Main result

The main result of this article is the following classification theorem.

Theorem 5.1. *Let M be a proper rectifying space-like submanifold of the pseudo-Euclidean m-space $\mathbb{E}_i^m$ with index $i > 0$. If* codim $M \geq 2$, *then one of the following four cases occurs:*

(a) *There exist a positive number c and local coordinate systems $\{s, u_2, \ldots, u_n\}$ on M such that the immersion of M in $\mathbb{E}_i^m$ is given by*
$$\mathbf{x}(s, u_2, \ldots, u_n) = \sqrt{s^2 + c^2}\, Y(s, u_2, \ldots, u_n), \tag{5.1}$$
where $Y = Y(s, u_2, \ldots, u_n)$ defines a space-like submanifolds of the unit pseudo-sphere $S_i^{m-1}(1) \subset \mathbb{E}_i^m$ such that the induced metric g_Y of Y is given by
$$g_Y = \frac{c^2}{(s^2 + c^2)^2}ds^2 + \frac{s^2}{s^2 + c^2}\sum_{j,k=2}^{n} g_{jk}(u_2, \ldots, u_n)du_j du_k. \tag{5.2}$$

(b) *There exist local coordinate systems $\{s, u_2, \ldots, u_n\}$ on M such that the immersion of M in $\mathbb{E}_i^m$ is given by*
$$\mathbf{x}(s, u_2, \ldots, u_n) = sW(s, u_2, \ldots, u_n), \quad s \neq 0, \tag{5.3}$$
where $W = W(s, u_2, \ldots, u_n)$ lies in the unit pseudo-sphere $S_i^{m-1}(1) \subset \mathbb{E}_i^m$ such that W_s is a light-like normal vector field of M and the induced metric tensor of W is of the following degenerate form:
$$g_W = \sum_{j,k=2}^{n} g_{jk}(u_2, \ldots, u_n)du_j du_k \tag{5.4}$$
with positive definite (g_{jk}), $j, k = 2, \ldots, n$.

(c) *There exist a positive number c and local coordinate systems $\{s, u_2, \ldots, u_n\}$ on M such that the immersion of M in $\mathbb{E}_i^m$ is given by*
$$\mathbf{x}(s, u_2, \ldots, u_n) = \sqrt{s^2 - c^2}\, U(s, u_2, \ldots, u_n), \quad s^2 > c^2, \tag{5.5}$$
where $U = U(s, u_2, \ldots, u_n)$ lies in the unit pseudo-sphere $S_i^{m-1}(1) \subset \mathbb{E}_i^m$ such that the induced metric g_U of U is given by
$$g_U = \frac{-c^2}{(s^2 - c^2)^2}ds^2 + \frac{s^2}{s^2 - c^2}\sum_{j,k=2}^{n} g_{jk}(u_2, \ldots, u_n)du_j du_k. \tag{5.6}$$

(d) *There exist a positive number c and local coordinate systems $\{s, u_2, \ldots, u_n\}$ on M such that the immersion of M in $\mathbb{E}_i^m$ is given by*
$$\mathbf{x}(s, u_2, \ldots, u_n) = \sqrt{c^2 - s^2}\, V(s, u_2, \ldots, u_n), \quad c^2 > s^2, \tag{5.7}$$
where $V = V(s, u_2, \ldots, u_n)$ lies in the pseudo-hyperbolic space $H_{i-1}^{m-1}(-1) \subset \mathbb{E}_i^m$ for $i > 1$ (respectively, hyperbolic space $H^{m-1}(-1) \subset \mathbb{E}_1^m$ for $i = 1$) such that the induced metric g_V of V is given by
$$g_V = \frac{c^2}{(c^2 - s^2)^2}ds^2 + \frac{s^2}{c^2 - s^2}\sum_{j,k=2}^{n} g_{jk}(u_2, \ldots, u_n)du_j du_k. \tag{5.8}$$

Conversely, each of the four cases above gives rise to a proper rectifying space-like submanifold of $\mathbb{E}^m_i$.

Proof. Assume that M is a proper rectifying space-like submanifold of $\mathbb{E}^m_i$ with $m \geq 2 + \dim M$. Then we have $\mathbf{x}^T \neq 0$ and $\mathbf{x}^N \neq 0$. Thus we may put

$$\mathbf{x}^T = \rho e_1, \quad \rho = |\mathbf{x}^T| > 0, \tag{5.9}$$

where e_1 is a space-like unit vector field. We may extend e_1 to a local orthonormal frame $e_1, e_2, \ldots, e_n$ on M. Clearly, we have $\langle \mathbf{x}, e_j \rangle = 0$ for $j = 2, \ldots, n$.

Define the connection forms $\omega_i^j, i, j = 1, \ldots, n$, by

$$\nabla_X e_i = \sum_{j=1}^{n} \omega_i^j(X) e_j, \quad i = 1, \ldots, n, \tag{5.10}$$

where ∇ is the Levi-Civita connection of M.

For $j, k = 2, \ldots, n$, we find

$$0 = e_k \langle \mathbf{x}, e_j \rangle = \delta_{jk} + \langle \mathbf{x}, \nabla_{e_k} e_j \rangle + \langle \mathbf{x}, h(e_j, e_k) \rangle = \delta_{jk} + \langle \mathbf{x}, \nabla_{e_k} e_j \rangle, \tag{5.11}$$

where we have applied (2.11) from Definition 2.1, (2.3) and (3.8).

Since $h(X, Y)$ is symmetric in X and Y, we derive from (5.10) and (5.11) that

$$\omega_j^1(e_k) = \omega_k^1(e_j), \quad j, k = 2, \ldots, n. \tag{5.12}$$

It follows from (5.10), (5.12) and the Frobenius theorem that the distribution $\mathcal{D}$ spanned by $e_2, \ldots, e_n$ is an integrable distribution.

On the other hand, the distribution $\mathcal{D}^\perp = \mathrm{Span}\,\{e_1\}$ is also integrable since it is of rank one. Therefore, there exists a local coordinate system $\{s, u_2, \ldots, u_n\}$ on M such that

$$e_1 = \frac{\partial}{\partial s} \quad \textit{and} \quad \mathcal{D} = \mathrm{Span}\left\{\frac{\partial}{\partial u_2}, \ldots, \frac{\partial}{\partial u_n}\right\}.$$

Obviously, it follows from (5.9) that $\rho = \langle \mathbf{x}, e_1 \rangle$. Now, by taking the derivative of ρ with respect to e_j for $j = 1, \ldots, n$ and using (2.3) and (3.8), we find

$$e_j \rho = \delta_{1j} + \langle \mathbf{x}, h(e_1, e_j) \rangle. \tag{5.13}$$

After combining (2.11) and (5.13) we find $e_1 \rho = 1$ and $e_2 \rho = \cdots = e_n \rho = 0$. Therefore we have

$$\rho = \rho(s), \quad \rho'(s) = 1$$

which imply

$$\rho(s) = s + b. \tag{5.14}$$

for some real number b. Consequently, after applying a suitable translation on s if necessary, we obtain $\rho = s$. Consequently, (5.9) implies that the position vector field satisfies

$$\mathbf{x} = s e_1 + \mathbf{x}^N. \tag{5.15}$$

Moreover, since M is a proper rectifying submanifold, Lemma 3.3 implies that $\left\langle \mathbf{x}^N, \mathbf{x}^N \right\rangle$ is constant on M. Therefore we find

$$\langle \mathbf{x}, \mathbf{x} \rangle = \begin{cases} s^2 + c^2, & \text{if } \left\langle \mathbf{x}^N, \mathbf{x}^N \right\rangle > 0, \\ s^2, & \text{if } \left\langle \mathbf{x}^N, \mathbf{x}^N \right\rangle = 0, \\ s^2 - c^2, & \text{if } \left\langle \mathbf{x}^N, \mathbf{x}^N \right\rangle < 0, \end{cases} \tag{5.16}$$

where c is a positive number.

Now, we divide the proof of the theorem into three cases.

Case (1): $\langle \mathbf{x}, \mathbf{x} \rangle = s^2 + c^2$ *with* $c > 0$. In this case, we may put

$$\mathbf{x}(s, u_2, \ldots, u_n) = \sqrt{s^2 + c^2}\, Y(s, u_2, \ldots, u_n), \tag{5.17}$$

for some $\mathbb{E}_i^m$-valued function $Y = Y(s, u_2, \ldots, u_n)$ satisfying $\langle Y, Y\rangle = 1$. Therefore the image of Y lies in the pseudo-sphere $S_i^{m-1}(1) \subset \mathbb{E}_i^{m-1}$. It follows from (5.17) that

$$\begin{aligned}\frac{\partial \mathbf{x}}{\partial s} &= \frac{s}{\sqrt{s^2+c^2}}Y + \sqrt{s^2+c^2}\,Y_s,\\ \frac{\partial \mathbf{x}}{\partial u_j} &= \sqrt{s^2+c^2}\,Y_{u_j}, \quad j = 2, \ldots, n.\end{aligned} \tag{5.18}$$

Using (5.18) together with the fact that $e_1 = \partial\mathbf{x}/\partial s$ is a unit vector field orthogonal to the distribution $\mathcal{D}$, we derive that

$$\langle Y_s, Y_s\rangle = \frac{c^2}{(s^2+c^2)^2}, \quad \langle Y_s, Y_{u_j}\rangle = 0, \quad j = 2, \ldots, n. \tag{5.19}$$

Therefore the metric tensor g_Y of Y induced from $S_i^{m-1}(1)$ takes the following form:

$$g_Y = \frac{c^2}{(s^2+c^2)^2}ds^2 + \frac{s^2}{s^2+c^2}\sum_{j,k=2}^{n} g_{jk}(s, u_2, \ldots, u_n)du_j du_k, \tag{5.20}$$

where (g_{jk}) is positive definite. In particular, (5.17) and (5.20) show that the submanifold defined by Y is also space-like.

Now, by applying (5.18) and (5.20) we know that the metric tensor g of M is of the form:

$$g = ds^2 + s^2\sum_{j,k=2}^{n} g_{jk}(s, u_2, \ldots, u_n)du_j du_k. \tag{5.21}$$

After a straight-forward long computation we find from (5.21) that the Levi-Civita connection of M satisfies

$$\begin{aligned}&\nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s} = 0,\\ &\nabla_{\frac{\partial}{\partial u_j}}\frac{\partial}{\partial s} = \frac{1}{s}\frac{\partial}{\partial u_j} + \frac{1}{2}\sum_{k=2}^{n}\left(\sum_{t=2}^{n} g^{kt}\frac{\partial g_{jt}}{\partial s}\right)\frac{\partial}{\partial u_k}, \quad j = 2, \ldots, n,\end{aligned} \tag{5.22}$$

where (g^{jk}) is the inverse matrix of (g_{ij}). Because M is a proper rectifying space-like submanifold of $\mathbb{E}_i^m$, it follows from Theorem 4.1 that

$$\nabla_{\frac{\partial}{\partial u_j}}\mathbf{x}^T = \frac{\partial}{\partial u_j}, \quad j = 2, \ldots, n. \tag{5.23}$$

Therefore, after applying (4.7), (5.22) and (5.23) we obtain

$$\sum_{t=2}^{n} g^{kt}\frac{\partial g_{jt}}{\partial s} = 0, \quad j, k = 2, \ldots, n. \tag{5.24}$$

Because (g^{jk}) is positive definite, system (5.24) implies

$$\frac{\partial g_{jk}}{\partial s} = 0, \quad j, t = 2, \ldots, n.$$

Therefore (5.31) must take the form of (5.4). Consequently, (5.20) reduces to (5.2).

Conversely, let us consider a space-like submanifold M of $\mathbb{E}_i^m$ defined by (5.1) satisfying $\langle Y, Y\rangle = 1$ such that the metric tensor g_Y is given by (5.2). Then we obtain (5.18) and (5.19) from (5.1). It follows from (5.2), (5.18) and (5.19) that the metric tensor g of M is given by

$$g = ds^2 + s^2\sum_{j,k=2}^{n} g_{jk}(u_2, \ldots, u_n)du_j du_k. \tag{5.25}$$

Now, it is straight-forward to verify from (5.25) that the Levi-Civita connection of M satisfies

$$\nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s} = 0, \quad \nabla_{\frac{\partial}{\partial u_j}}\frac{\partial}{\partial s} = \frac{1}{s}\frac{\partial}{\partial u_j}, \quad j = 2, \ldots, n. \tag{5.26}$$

Since $\langle Y, Y\rangle = 1$, (5.1) implies $\langle \mathbf{x}, Y_{u_j}\rangle = 0$ for $j = 2, \dots, n$. Thus we find from (5.18) that

$$\langle \mathbf{x}, \mathbf{x}_{u_j}\rangle = 0, \quad j = 2, \dots, n. \tag{5.27}$$

Therefore, we obtain $\mathbf{x}^T = s\frac{\partial}{\partial s}$. Now, by applying (5.26) it is easy to verify that $\mathbf{x}^T$ is a concurrent vector field on M. Moreover, it is direct to show that the normal component of $\mathbf{x}$ is given by

$$\mathbf{x}^N = \frac{c^2}{\sqrt{s^2 + c^2}} Y - s\sqrt{s^2 + c^2}\, Y_s,$$

which is alway non-zero everywhere on M. Consequently, the immersion defined by case (a) gives rise to a proper rectifying space-like submanifold of $\mathbb{E}^m_i$.

Case (2): $\langle \mathbf{x}, \mathbf{x}\rangle = s^2$, $s \neq 0$. In this case, $\mathbf{x}^N$ is a light-like normal vector field of M.

We put

$$\mathbf{x}(s, u_2, \dots, u_n) = s\, W(s, u_2, \dots, u_n), \quad s \neq 0, \tag{5.28}$$

for some $\mathbb{E}^m_i$-valued function $W = W(s, u_2, \dots, u_n)$ satisfying $\langle W, W\rangle = 1$. Therefore the image of W lies in the pseudo-sphere $S^{m-1}_i(1) \subset \mathbb{E}^{m-1}_i$.

It follows from (5.28) that

$$\frac{\partial \mathbf{x}}{\partial s} = W + sW_s, \quad \frac{\partial \mathbf{x}}{\partial u_j} = sW_{u_j}, \quad j = 2, \dots, n. \tag{5.29}$$

Using (5.29), $\langle W, W\rangle = 1$ and the fact that $e_1 = \partial \mathbf{x}/\partial s$ is a unit vector field orthogonal to the distribution $\mathcal{D}$, we derive that

$$\langle W_s, W_s\rangle = 0, \quad \langle W_s, W_{u_j}\rangle = 0, \quad j = 2, \dots, n. \tag{5.30}$$

If we put $g_{jk} = \langle W_{u_j}, W_{u_k}\rangle$, then it follows from (5.29) and (5.30) that the metric tensor g_Y of W is a generate one given by

$$g_W = \sum_{j,k=2}^{n} g_{ij}(s, u_2, \dots, u_n) du_j du_k. \tag{5.31}$$

Then it follows from (5.28) and (5.31) that the induced metric g of M is given by

$$g = ds^2 + s^2 \sum_{j,k=2}^{n} g_{jk}(s, u_2, \dots, u_n) du_j du_k. \tag{5.32}$$

Since M is a proper rectifying space-like submanifold of $\mathbb{E}^m_i$, it follows from Theorem 4.1 that $\mathbf{x}^T$ is a concurrent vector field. Therefore, we may apply the same argument as in Case (1) to conclude that $\partial g_{jk}/\partial s = 0$ for $j, t = 2, \dots, n$. Therefore (5.31) must take the form of (5.4).

Conversely, let us consider an immersion $x : M \to \mathbb{E}^m_i$ of a Riemannian n-manifold M into $\mathbb{E}^m_i$ given by

$$\mathbf{x}(s, u_2, \dots, u_n) = sW(s, u_2, \dots, u_n), \quad \langle W, W\rangle = 1, \quad s \neq 0, \tag{5.33}$$

such that W_s is a light-like normal vector field and the metric tensor of W is of the following degenerate form:

$$g_W = \sum_{j,k=2}^{n} g_{jk}(u_2, \dots, u_n) du_j du_k \tag{5.34}$$

with positive definite matrix (g_{jk}), $j, k = 2, \dots, n$. Then it follows from (5.33) and (5.34) that the induced metric g of M is given by

$$g = ds^2 + s^2 \sum_{j,k=2}^{n} g_{jk}(u_2, \dots, u_n) du_j du_k. \tag{5.35}$$

From (5.34) we get

$$\mathbf{x}_s = W + sW_s, \quad \mathbf{x}_{u_j} = sW_{u_j}, \quad j = 2, \dots, n. \tag{5.36}$$

Thus we find from (5.33) and (5.36) that

$$\mathbf{x} = s\mathbf{x}_s - s^2 W_s. \tag{5.37}$$

Because W_s is a light-like normal vector field and $\mathbf{x}_s$ is tangent to M, we obtain from (5.37) that

$$\mathbf{x}^T = s\mathbf{x}_s \ \ and \ \ \mathbf{x}^N = -s^2 W_s \neq 0. \tag{5.38}$$

Now, we may derive from (5.35) and (5.38) as before that $\mathbf{x}^T$ is a concurrent vector field on M. Consequently, M is a rectifying space-like submanifold of $\mathbb{E}^m_i$ according to Theorem 4.1. This gives Case (b) of the theorem.

Case (3): $\langle \mathbf{x}, \mathbf{x} \rangle = s^2 - c^2 \neq 0$. By applying a method similar to Case (1), we will obtain either Case (c) or Case (d) according to $s^2 > c^2$ or $s^2 < c^2$, respectively. □

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