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Addendum to: Differential Geometry of Rectifying Submanifolds

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ABSTRACT

We point out that the proof of Theorem 4.2 of [B.-Y. Chen, Differential geometry of rectifying submanifolds, Int. Electron. J. Math. 9 (2016), no. 2, 1–8] holds only for rectifying submanifolds with codimension ≥ 2. For rectifying submanifolds of codimension one, we classify rectifying hypersurfaces in a Euclidean space.

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1. Introduction

Let M be a Riemannian manifold isometrically immersed in the Euclidean m-space $\mathbb{E}^m$. Denote by h and A the second fundamental form and the shape operator of M in $\mathbb{E}^m$, respectively. For a point $p \in M$, the *first normal subspace*, $\operatorname{Im} h_p$, of M at p is the subspace defined by

$$\operatorname{Im} h_p = \operatorname{Span}\{h(X, Y) : X, Y \in T_pM\},$$

where T_pM denotes the tangent space of M^n at p.

We recall the following definitions from [2].

Definition 1.1. For a submanifold M of $\mathbb{E}^m$ and a point $p \in M$, the orthogonal complement of $\operatorname{Im} \sigma_p$ in $T_p\mathbb{E}^m$ is called the *rectifying space of M* at p.

Definition 1.2. A submanifold M of $\mathbb{E}^m$ is called a *rectifying submanifold* if the position vector field $\mathbf{x}$ of M, relative to the origin $o \in \mathbb{E}^m$, always lies in its rectifying space. In other words, M is a rectifying submanifold if and only if $\langle \mathbf{x}(p), \operatorname{Im} h_p\rangle = 0$ holds at every $p \in M$.

Definition 1.3. A non-trivial vector field Z on a Riemannian manifold M is called *concurrent* if it satisfies $\nabla_X Z = X$ for any vector $X \in TM$, where ∇ is the Levi-Civita connection of M.

For a submanifold of $\mathbb{E}^m$, there exists a natural orthogonal decomposition of the position vector field $\mathbf{x}$ of M at each point; namely,

$$\mathbf{x} = \mathbf{x}^T + \mathbf{x}^N, \tag{1.1}$$

where $\mathbf{x}^T$ and $\mathbf{x}^N$ denote the tangential and normal components of $\mathbf{x}$, respectively.

Definition 1.4. A rectifying submanifold M of $\mathbb{E}^m$ is called *proper* if it satisfies $\mathbf{x} \neq \mathbf{x}^T$ and $\mathbf{x} \neq \mathbf{x}^N$ almost everywhere.

The following result was proved in [2].

Theorem 1.1. *If the position vector field $\mathbf{x}$ of a submanifold M in $\mathbb{E}^m$ satisfies $\mathbf{x}^N \neq 0$, then M is a proper rectifying submanifold if and only if $\mathbf{x}^T$ is a concurrent vector field on M.*

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2. A remark on Theorem 4.2 of [2]

First we want to point out that one requires the condition $m \geq 2 + \dim M$ in the proof of Theorem 4.2 of [2]. However, this condition was missing in the statement of theorem in [2]. Hence Theorem 4.2 of [2] shall be restated as the following.

Theorem 2.1. *Let M is a proper rectifying submanifold of $\mathbb{E}^m$. If $m \geq 2 + \dim M$, then with respect to some suitable local coordinate systems $\{s, u_2, \ldots, u_n\}$ on M the immersion x of M in $\mathbb{E}^m$ takes the form:*

$$x(s, u_2, \ldots, u_n) = \sqrt{s^2 + c^2}\, Y(s, u_2, \ldots, u_n), \quad \langle Y, Y \rangle = 1, \ c > 0, \tag{2.1}$$

such that the metric tensor g_Y of the spherical submanifold defined by Y satisfies

$$g_Y = \frac{c^2}{(s^2 + c^2)^2} ds^2 + \frac{s^2}{s^2 + c^2} \sum_{i,j=2}^{n} g_{ij}(u_2, \ldots, u_n) du_i du_j. \tag{2.2}$$

Conversely, the immersion given by (2.1)-(2.2) defines a proper rectifying submanifold.

3. Classification of rectifying hypersurfaces

Now, we classify rectifying hypersurfaces.

Theorem 3.1. *A proper hypersurface M of $\mathbb{E}^{n+1}$ is a rectifying hypersurface if and only if M is an open portion of a hyperplane L of $\mathbb{E}^{n+1}$ with $o \notin L$, where o denotes the origin of $\mathbb{E}^{n+1}$.*

Proof. Let M be a rectifying proper hypersurface of $\mathbb{E}^{n+1}$. Then we have $\nabla_Z \mathbf{x}^T = Z$ for any $Z \in TM$. Combining this with (4.3) in [2] gives $A_{\mathbf{x}^N} = 0$ identically. Hence M a is totally geodesic hypersurface in $\mathbb{E}^{n+1}$. Consequently, M is an open portion of a hyperplane L of $\mathbb{E}^{n+1}$ (cf. [1, page 54]).

If the origin o of $\mathbb{E}^{n+1}$ lies in L, then the position vector field $\mathbf{x}$ of M is tangent to M at each point on M. Hence M is non-proper. Consequently, we must have $o \notin L$.

Conversely, suppose that M is an open portion of a hyperplane L such that $o \notin L$. Then it is clearly that M is a proper hypersurface. Let $\tilde{\nabla}$ denote the Levi-Civita connection of $\mathbb{E}^{n+1}$. Then we have

$$Z = \tilde{\nabla}_Z \mathbf{x} = \tilde{\nabla}_Z \mathbf{x}^T + \tilde{\nabla}_Z \mathbf{x}^N. \tag{3.1}$$

Since M is totally geodesic in $\mathbb{E}^{n+1}$, it follows from (3.1), the formula of Gauss and the formula Weingarten that $\nabla_Z \mathbf{x}^T = Z$. Therefore $\mathbf{x}^T$ is a concurrent vector field. Consequently, Theorem 1.1 implies that M is a rectifying hypersurface. □

The pseudo-Riemannian version of Theorem 3.1 holds as well.

Theorem 3.2. *A pseudo-Riemannian proper hypersurface M_t with index t in a pseudo-Euclidean space $\mathbb{E}^{n+1}_s$ with index s is a rectifying hypersurface if and only if M_t is an open portion of a pseudo-Euclidean hyerplance L_t of $\mathbb{E}^{n+1}_s$ with $o \notin L_t$, where o denotes the origin of $\mathbb{E}^{n+1}_s$.*

Proof. This can be proved in the same way as Theorem 3.1. □

References

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