

Chen Inequalities on Lightlike Hypersurface of a Lorentzian Manifold with Semi-Symmetric Metric Connection

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ABSTRACT

In this paper, we introduce k-Ricci curvature and k-scalar curvature on lightlike hypersurface of a Lorentzian manifold with semi-symmetric metric connection. Using these curvatures, we establish some inequalities for lightlike hypersurface of a Lorentzian manifold with semi-symmetric metric connection. Considering these inequalities, we obtain the relation between Ricci curvature and scalar curvature endowed with semi-symmetric metric connection.

Keywords: Chen inequality; lightlike hypersurface; Lorentzian manifold; semi-symmetric metric connection.

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1. Introduction

Hayden [17] introduced a semi-symmetric metric connection on a Riemannian manifold. Imai [20] gave basic properties of a hypersurface of a Riemannian manifold with semi-symmetric metric connection and get conformal equations of Gauss and Codazzi. Konar and Biswas [22] considered semi-symmetric metric connection on Lorentz manifold. They showed that the perfect fluid space time with a non-zero constant scalar curvature which admits a semi-symmetric metric connection whose Ricci tensor is zero has vanishing expansion scalar and acceleration vector.

In 1993, Chen [9] introduced a new Riemannian invariant for a Riemannian manifold M as follows:

$$\delta_M = \tau(p) - \text{inf}(K)(p) \quad (1.1)$$

where $\tau(p)$ is scalar curvature of M and

$$\text{inf}(K)(p) = \{\text{inf}K(\Pi) : K(\Pi) \text{ is a plane section of } T_pM\}.$$

In [5], Chen established a sharp inequality for submanifold in a real space form involving intrinsic invariants, namely the sectional curvature and the scalar curvature of the submanifold; and the main extrinsic invariant, namely the squared mean curvature.

Afterwards, Chen and some geometers studied similar problems for non-degenerate submanifolds of different spaces such as in [4, 6, 8, 26]. Later Mihai and Özgür in [23] studied Chen inequalities on submanifolds of real space forms endowed with semi-symmetric metric connection.

Gülbahar, Kılıç and Keleş introduced Chen-like inequalities and curvature invariants in lightlike geometry. Also, they established some inequalities between the extrinsic scalar curvatures and the intrinsic scalar curvatures [14]. In [15], they established some inequalities involving k-Ricci curvature, k-scalar curvature, the screen scalar curvature on a screen homothetic lightlike hypersurface of a Lorentzian manifold. Poyraz and Yaşar introduced k-Ricci curvature and k-scalar curvature on lightlike hypersurface of a Lorentzian product manifold with quarter-symmetric nonmetric connection and using these curvatures they established some

Chen-type inequalities for screen homothetic lightlike hypersurface of a Lorentzian product manifold with quarter-symmetric nonmetric connection [25].

In this paper, we study inequalities for screen homothetic lightlike hypersurface of a real space form $\widetilde{M}(c)$ of constant sectional curvature c , endowed with semi symmetric metric connection. Considering these inequalities, we obtain the relation between Ricci curvature and scalar curvature endowed with the semi symmetric metric connection.

2. Preliminaries

Let M be a hypersurface of a $(n + 1)$ -dimensional, $n > 1$, semi-Riemannian manifold \widetilde{M} with semi-Riemannian metric \tilde{g} of index $1 \leq \nu \leq n$. We consider

$$T_x M^\perp = \left\{ Y_x \in T_x \widetilde{M} \mid \tilde{g}_x(Y_x, X_x) = 0, \forall X_x \in T_x M \right\}$$

for any $x \in M$. Then we say that M is a *lightlike (null, degenerate) hypersurface* of \widetilde{M} or equivalently, the immersion

$$i : M \rightarrow \widetilde{M}$$

of M in \widetilde{M} is *lightlike (null, degenerate)* if $T_x M \cap T_x M^\perp \neq \{0\}$ at any $x \in M$.

An orthogonal complementary vector bundle of TM^\perp in TM is non-degenerate subbundle of TM named the *screen distribution* on M and denoted $S(TM)$. We have the following splitting into orthogonal direct sum:

$$TM = S(TM) \perp TM^\perp. \tag{2.1}$$

The subbundle $S(TM)$ is non-degenerate, so is $S(TM)^\perp$, and the following satisfies:

$$T\widetilde{M} = S(TM) \perp S(TM)^\perp, \tag{2.2}$$

where $S(TM)^\perp$ is the orthogonal complementary vector bundle to $S(TM)$ in $T\widetilde{M}|_M$.

Let $tr(TM)$ denote the complementary vector bundle of TM^\perp in $S(TM)^\perp$. Then we have

$$S(TM)^\perp = TM^\perp \oplus tr(TM). \tag{2.3}$$

Let U be a coordinate neighborhood in M and ξ be a basis of $\Gamma(TM^\perp|_U)$. Then there exists a basis N of $tr(TM)|_U$ satisfying the following conditions:

$$\tilde{g}(N, \xi) = 1,$$

and

$$\tilde{g}(N, N) = \tilde{g}(W, N) = 0, \quad \forall W \in \Gamma(S(TM)|_U).$$

The subbundle $tr(TM)$ is named a *lightlike transversal vector bundle* of M . We note that $tr(TM)$ is never orthogonal to TM . From (2.1), (2.2) and (2.3) we have

$$T\widetilde{M}|_M = S(TM) \perp (TM^\perp \oplus tr(TM)) = TM \oplus tr(TM), \tag{2.4}$$

[11, 16].

3. Semi-Symmetric Metric Connection

For $n > 1$, let \widetilde{M} be an $(n + 2)$ -dimensional differentiable manifold of class C^∞ and $\widetilde{\nabla}$ a linear connection in \widetilde{M} . The torsion tensor \widetilde{T} of $\widetilde{\nabla}$ is given by

$$\widetilde{T}(\widetilde{X}, \widetilde{Y}) = \widetilde{\nabla}_{\widetilde{X}} \widetilde{Y} - \widetilde{\nabla}_{\widetilde{Y}} \widetilde{X} - [\widetilde{X}, \widetilde{Y}], \quad \forall \widetilde{X}, \widetilde{Y} \in \Gamma(T\widetilde{M})$$

and have type (1, 2). When the torsion tensor \tilde{T} satisfies

$$\tilde{T}(\tilde{X}, \tilde{Y}) = \tilde{\pi}(\tilde{Y})\tilde{X} - \tilde{\pi}(\tilde{X})\tilde{Y}$$

for a 1-form $\tilde{\pi}$, the connection $\tilde{\nabla}$ is said to be *semi-symmetric* (see [27]).

Let us consider a semi-Riemannian metric \tilde{g} of index ν with $1 \leq \nu \leq n + 1$ in \tilde{M} and $\tilde{\nabla}$ satisfying

$$\tilde{\nabla}\tilde{g} = 0.$$

A linear connection of this type is called a *metric connection* (see [23]).

We assume that the semi-Riemannian manifold \tilde{M} admits a semi-symmetric metric connection which is given by

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \overset{\circ}{\nabla}_{\tilde{X}}\tilde{Y} + \tilde{\pi}(\tilde{Y})\tilde{X} - \tilde{g}(\tilde{X}, \tilde{Y})\tilde{Q} \tag{3.1}$$

for arbitrary vector fields \tilde{X} and \tilde{Y} of \tilde{M} , where $\overset{\circ}{\nabla}$ denotes the Levi-Civita connection with respect to the semi-Riemannian metric \tilde{g} , $\tilde{\pi}$ is a 1-form and \tilde{Q} is the vector field defined by

$$\tilde{g}(\tilde{Q}, \tilde{X}) = \tilde{\pi}(\tilde{X})$$

for an arbitrary vector field \tilde{X} of \tilde{M} (see [13] and [27]).

The *Gauss formula* with respect to the induced connection ∇ on the lightlike hypersurface from the semi-symmetric metric connection $\tilde{\nabla}$ is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + m(X, Y)N \tag{3.2}$$

for arbitrary vector fields X and Y of M , where m is a tensor of type (0, 2) of the lightlike hypersurface of M [28].

On the other hand, denoting the projection of TM on $S(TM)$ with respect to the decomposition (2.1) by P , one has the *Weingarten formula* with respect to the semi-symmetric connection which is given by

$$\nabla_X PY = \overset{*}{\nabla}_X PY + D(X, PY)\xi, \tag{3.3}$$

where $\overset{*}{\nabla}_X PY$ belongs to $\Gamma(S(TM))$ and D is 1-form on M .

The curvature tensor $\overset{\circ}{R}$ with respect to $\overset{\circ}{\nabla}$ on real space form $\tilde{M}(c)$ is defined by

$$\overset{\circ}{R}(X, Y, Z, W) = c\{g(X, W)g(Y, Z) - g(Y, W)g(X, Z)\}. \tag{3.4}$$

Using (3.1), for any vector fields $X, Y, Z, W \in \Gamma(TM)$ and (0, 2) tensor field α which defined by

$$\alpha(X, Y) = (\overset{\circ}{\nabla}_X \pi)Y - \pi(X)\pi(Y) + \frac{1}{2}\pi(Q)g(X, Y) \tag{3.5}$$

we have relation between the curvature tensor $\overset{\circ}{R}$ with respect to the Levi-Civita connection $\overset{\circ}{\nabla}$ and the curvature tensor \tilde{R} with respect to the semi-symmetric metric connection $\tilde{\nabla}$ given by

$$\begin{aligned} \tilde{R}(X, Y, Z, W) &= \overset{\circ}{R}(X, Y, Z, W) - \alpha(Y, Z)g(X, W) + \alpha(X, Z)g(Y, W) \\ &\quad - \alpha(X, W)g(Y, Z) + \alpha(Y, W)g(X, Z), \end{aligned} \tag{3.6}$$

[19].

Moreover, Gauss-Codazzi equations with respect to the semi-symmetric metric connection $\tilde{\nabla}$ on \tilde{M} can be written as [28]

$$\begin{aligned} R(X, Y, Z, PW) &= \tilde{R}(X, Y, Z, PW) - m(X, Z)D(Y, PW) + m(Y, Z)D(X, PW) \\ &\quad - \{m(X, Z)\eta(Y) - m(Y, Z)\eta(X)\}\pi(PW), \end{aligned} \tag{3.7}$$

$$\begin{aligned} \tilde{g}(\tilde{R}(X, Y)Z, \xi) &= \pi(Y)m(X, Z) - \pi(X)m(Y, Z) + (\nabla_X m)(Y, Z) - (\nabla_Y m)(X, Z) \\ &\quad + m(Y, Z)(\tau(X) - \mu\eta(X)) - m(X, Z)(\tau(Y) - \mu\eta(Y)), \end{aligned} \tag{3.8}$$

and

$$\tilde{g}(\tilde{R}(X, Y)Z, N) = g(R(X, Y)Z, N), \tag{3.9}$$

for any vector fields $X, Y, Z, W \in \Gamma(TM)$.

From (3.4), (3.6) and (3.7), we have

$$\begin{aligned} R(X, Y, Z, PW) &= c\{g(Y, Z)g(X, PW) - g(X, Z)g(Y, PW)\} \\ &\quad - \alpha(Y, Z)g(X, PW) + \alpha(X, Z)g(Y, PW) \\ &\quad - \alpha(X, W)g(Y, Z) + \alpha(Y, W)g(X, Z) \\ &\quad - m(X, Z)D(Y, PW) + m(Y, Z)D(X, PW) \\ &\quad - \{m(X, Z)\eta(Y) - m(Y, Z)\eta(X)\}\pi(PW). \end{aligned} \tag{3.10}$$

Denote by λ the trace of α .

Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold (\tilde{M}, \tilde{g}) . Then M is named totally umbilical lightlike hypersurface if there exists a smooth function such that

$$m(X, Y)_p = Hg_p(X, Y), \quad X, Y \in \Gamma(T_pM) \tag{3.11}$$

for any coordinate neighborhood U and $X, Y \in \Gamma(TM|_U)$, where $H \in R$. If every points of M is umbilical, the lightlike hypersurface M is named totally umbilical in \tilde{M} [11]. If $m = 0$, then the lightlike hypersurface M is named totally geodesic in \tilde{M} .

The mean curvature μ of M with respect to an orthonormal basis $\{e_1, \dots, e_n\}$ of $\Gamma(S(TM))$ is defined by

$$\mu = \frac{1}{n}tr(m) = \frac{1}{n} \sum_{i=1}^n \varepsilon_i m(e_i, e_i), \quad g(e_i, e_i) = \varepsilon_i. \tag{3.12}$$

A lightlike hypersurface (M, g) of a semi-Riemannian manifold (\tilde{M}, \tilde{g}) is called *screen locally conformal* if the shape operators A_N and A_ξ^* of M and $S(TM)$, respectively, are related by

$$A_N = \varphi A_\xi^*, \tag{3.13}$$

where φ is a non-vanishing smooth function on a neighborhood U on M . In particular, if φ is a non-zero constant, M is called screen homothetic [12].

Let M be a two-dimensional non-degenerate plane. The sectional curvature at $p \in M$ is given by

$$K_{ij} = \frac{g(R(e_j, e_i)e_i, e_j)}{g(e_i, e_i)g(e_j, e_j) - g(e_i, e_j)^2} \tag{3.14}$$

[12].

Let $p \in M$ and ξ be null vector of T_pM . A plane Π of T_pM is said to be null plane if it contains ξ and e_i such that $g(\xi, e_i) = 0$ and $g(e_i, e_i) = \varepsilon_i = \pm 1$. One defines the null sectional curvature of Π by

$$K_i^{null} = \frac{g(R_p(e_i, \xi)\xi, e_i)}{g_p(e_i, e_i)}$$

[2].

We denote the Ricci tensor of \tilde{M} with \tilde{Ric} and the induced Ricci type tensor of M with $R^{(0,2)}$. Then, \tilde{Ric} and $R^{(0,2)}$ are given by

$$\begin{aligned} \tilde{Ric}(X, Y) &= trace\{Z \rightarrow \tilde{R}(Z, X)Y\}, \quad \forall X, Y \in \Gamma(T\tilde{M}), \\ R^{(0,2)}(X, Y) &= trace\{Z \rightarrow R(Z, X)Y\}, \quad \forall X, Y \in \Gamma(TM), \end{aligned} \tag{3.15}$$

where

$$R^{(0,2)}(X, Y) = \sum_{i=1}^n \varepsilon_i g(R(e_i, X)Y, e_i) + \tilde{g}(R(\xi, X)Y, N) \tag{3.16}$$

for the quasi-orthonormal frame $\{e_1, \dots, e_n, \xi\}$ of T_pM .

Scalar curvature τ is defined

$$\tau(p) = \sum_{i,j=1}^n K_{ij} + \sum_{i=1}^n K_i^{null} + K_{iN}, \tag{3.17}$$

where $K_{iN} = \tilde{g}(R(\xi, e_i)e_i, N)$ for $i \in \{1, \dots, n\}$ [10].

If $\dim(M) > 2$ and

$$Ric(X, Y) = kg(X, Y), \tag{3.18}$$

then M is an Einstein manifold. For $\dim(M) = 2$, any M is Einstein but k in (3.18) is not necessarily constant [12].

4. Chen Ricci Inequality

In this section, we use the same notations and terminologies as in [14].

Let M be an $(n + 1)$ -dimensional lightlike hypersurface of a Lorentzian manifold \tilde{M} with a semi-symmetric metric connection. $\{e_1, \dots, e_n, \xi\}$ and $\{e_1, \dots, e_n\}$ are basis of $\Gamma(TM)$ and an orthonormal basis of $\Gamma(S(TM))$, respectively. Similarly, for $k \leq n$, $\pi_{k,\xi} = sp\{e_1, \dots, e_k, \xi\}$ and $\pi_k = sp\{e_1, \dots, e_k\}$ are $(k + 1)$ -dimensional degenerate plane section and $\pi_k = sp\{e_1, \dots, e_k\}$ is k -dimensional non-degenerate plane section, respectively. For a unit vector $X \in \Gamma(TM)$, the k -degenerate Ricci curvature and the k -Ricci curvature are defined by

$$Ric_{\pi_{k,\xi}}(X) = R^{(0,2)}(X, X) = \sum_{j=1}^k g(R(e_j, X)X, e_j) + \tilde{g}(R(\xi, X)X, N), \tag{4.1}$$

$$Ric_{\pi_k}(X) = R^{(0,2)}(X, X) = \sum_{j=1}^k g(R(e_j, X)X, e_j), \tag{4.2}$$

respectively [14]. Also for $p \in M$, k -degenerate scalar curvature and k -scalar curvature are determined by

$$\tau_{\pi_{k,\xi}}(p) = \sum_{i,j=1}^k K_{ij} + \sum_{i=1}^k K_i^{null} + K_{iN}, \tag{4.3}$$

$$\tau_{\pi_k}(p) = \sum_{i,j=1}^k K_{ij}, \tag{4.4}$$

respectively [14]. For $k = n$, $\pi_n = sp\{e_1, \dots, e_n\} = \Gamma(S(TM))$, we have the screen Ricci curvature and the screen scalar curvature given by

$$Ric_{S(TM)}(e_1) = Ric_{\pi_n}(e_1) = \sum_{j=1}^n K_{1j} = K_{12} + \dots + K_{1n}, \tag{4.5}$$

and

$$\tau_{S(TM)} = \sum_{i,j=1}^n K_{ij}, \tag{4.6}$$

respectively [14].

Using (3.10) we obtain

$$\tau_{S(TM)}(p) = n(n - 1)c - 2(n - 1)\lambda + \sum_{i,j=1}^n m_{ii}D_{jj} - m_{ij}D_{ji}, \tag{4.7}$$

where λ is the trace of α and $m_{ij} = m(e_i, e_j)$, $D_{ij} = D(e_i, e_j)$ for $i, j \in \{1, \dots, n\}$.

Let $\widetilde{M}(c)$ be a Lorentzian space form and M be a screen homothetic lightlike hypersurface of an $(n + 2)$ -dimensional $\widetilde{M}(c)$. Using (3.6)-(3.10) we get the following equations:

$$\tau_{S(TM)}(p) = n(n - 1)c - 2(n - 1)\lambda + \varphi n^2 \mu^2 - \varphi \sum_{i,j=1}^n (m_{ij})^2, \tag{4.8}$$

$$\begin{aligned} \sum_{i=1}^n K_i^{null} &= \sum_{i=1}^n R(e_i, \xi, \xi, e_i) \\ &= \sum_{i=1}^n \widetilde{R}(e_i, \xi, \xi, e_i) \\ &= \sum_{i=1}^n -\alpha(\xi, \xi) = -n\alpha(\xi, \xi), \end{aligned} \tag{4.9}$$

$$\begin{aligned} \sum_{i=1}^n K_i^N &= \sum_{i=1}^n R(\xi, e_i, e_i, N) \\ &= \sum_{i=1}^n \widetilde{R}(\xi, e_i, e_i, N) \\ &= \sum_{i=1}^n (c - \alpha(\xi, N) - \alpha(e_i, e_i)) \\ &= nc - n\alpha(\xi, N) - \lambda. \end{aligned} \tag{4.10}$$

From (3.17), (4.8), (4.9) and (4.10), we get the induced scalar curvature $\tau(p)$ of M as following:

$$\tau(p) = n^2c - 2(n + 1)\lambda + \varphi n^2 \mu^2 - \varphi \sum_{i,j=1}^n (m_{ij})^2 - n(\alpha(\xi, \xi) + \alpha(\xi, N)). \tag{4.11}$$

Using (4.11) we obtain the following :

Theorem 4.1. *Let M be an $(n + 1)$ -dimensional screen homothetic lightlike hypersurface with $\varphi > 0$ of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c , endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. Then we have*

$$\frac{1}{\varphi} (\tau(p) - n^2c + 2(n + 1)\lambda + n(\alpha(\xi, \xi) + \alpha(\xi, N))) \leq n^2 \mu^2 \tag{4.12}$$

The equality of (4.12) holds for $p \in M$ if and only if p is a totally geodesic point..

Lemma 4.1. [26] *Let a_1, a_2, \dots, a_n , be n -real number ($n > 1$), then*

$$\frac{1}{n} \left(\sum_{i=1}^n a_i \right)^2 \leq \sum_{i=1}^n a_i^2$$

with equality iff $a_1 = a_2 = \dots = a_n$.

Theorem 4.2. *Let M be an $(n + 1)$ -dimensional screen homothetic lightlike hypersurface with $\varphi > 0$ of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c , endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. Then we have*

$$\frac{1}{\varphi} (\tau(p) - n^2c + 2(n + 1)\lambda + n(\alpha(\xi, \xi) + \alpha(\xi, N))) \leq n(n - 1)\mu^2. \tag{4.13}$$

For $p \in M$ the equality of (4.13) satisfies iff p is a totally umbilical point.

Proof. Using Lemma 4.1 one derives

$$n\mu^2 \leq \sum_{i=1}^n (m_{ii})^2. \tag{4.14}$$

After substituting (4.14) in (4.11) we find (4.13). For $p \in M$ the equality of (4.13) satisfies iff

$$m_{11} = \dots = m_{nn}.$$

Thus p is a totally umbilical point. □

If the sectional curvature is screen homothetic, then the sectional curvature of lightlike hypersurface is symmetric. One defines the screen scalar curvature $r_{S(TM)}$

$$r_{S(TM)}(p) = \sum_{1 \leq i < j \leq n} K_{ij} = \frac{1}{2} \sum_{i,j=1}^n K_{ij} = \frac{1}{2} \tau_{S(TM)}(p). \tag{4.15}$$

By using (4.8), the equality (4.15) can be rewritten as follows:

$$2r_{S(TM)}(p) = n(n-1)c - 2(n-1)\lambda + \varphi n^2 \mu^2 - \sum_{i,j=1}^n (m_{ij})^2. \tag{4.16}$$

Theorem 4.3. *Let M be an $(n+1)$ -dimensional screen homothetic lightlike hypersurface with $\varphi > 0$ of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c , endowed with a semi-symmetric metric connection $\widetilde{\nabla}$ such that the vector field P is tangent to M . Then, the following statements are true.*

(i) For $X \in S^1(TM) = \{X \in S(TM) : \langle X, X \rangle = 1\}$

$$Ric_{S(TM)}(X) \leq \frac{1}{4} \varphi n^2 \mu^2 + (n-1)c - (2n-3)\lambda + (n-2)\alpha(X, X). \tag{4.17}$$

(ii) The equality case of (4.17) is satisfied by $X \in T_p^1(M)$ iff

$$\begin{aligned} m(X, Y) &= 0, \text{ for all } Y \in T_p(M) \text{ orthogonal to } X, \\ m(X, X) &= \frac{n}{2} \mu. \end{aligned} \tag{4.18}$$

(iii) The equality case of (4.17) holds for all $X \in T_p^1(M)$ iff either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.

Proof. From (4.16), we get

$$\begin{aligned} \frac{1}{4} \varphi n^2 \mu^2 &= r_{S(TM)}(p) - \frac{n(n-1)}{2} c + (n-1)\lambda + \frac{1}{4} \varphi (m_{11} - m_{22} - \dots - m_{nn})^2 \\ &\quad + \varphi \sum_{j=2}^n (m_{1j})^2 - \varphi \sum_{2 \leq i < j \leq n} m_{ii} m_{jj} - (m_{ij})^2. \end{aligned} \tag{4.19}$$

Using (3.10) we obtain

$$\varphi \sum_{2 \leq i < j \leq n} m_{ii} m_{jj} - (m_{ij})^2 = \sum_{2 \leq i < j \leq n} K_{ij} - \frac{(n-2)(n-1)}{2} c + (n-2)(\lambda - \alpha(e_1, e_1)). \tag{4.20}$$

From (4.19) and (4.20), we have

$$\begin{aligned} Ric_{S(TM)}(e_1) &= \frac{1}{4} \varphi n^2 \mu^2 + (n-1)c - (2n-3)\lambda - \frac{1}{4} \varphi (m_{11} - m_{22} - \dots - m_{nn})^2 \\ &\quad - \varphi \sum_{j=2}^n (m_{1j})^2 + (n-2)\alpha(e_1, e_1). \end{aligned} \tag{4.21}$$

If we take $e_1 = X$ like any vector of $T_p^1(M)$ in (4.21) one gets (4.17).

Equality holds in (4.17) for $X \in T_p^1(M)$ iff

$$m_{12} = m_{13} = \dots = m_{1n} = 0 \text{ and } m_{11} = m_{22} + \dots + m_{nn}, \tag{4.22}$$

which is equivalent to (4.18).

Now we prove the statement (iii). Assuming the equality in (4.17) for all $X \in T_p^1(M)$, in view of (4.22), we have

$$m_{ij} = 0, \quad i \neq j. \tag{4.23}$$

$$2m_{ii} = m_{11} + m_{22} + \dots + m_{nn}, \quad i \in \{1, \dots, n\}. \tag{4.24}$$

From (4.24), we have $2m_{11} = 2m_{22} = \dots = 2m_{nn} = m_{11} + m_{22} + \dots + m_{nn}$, which implies that

$$(n - 2)(m_{11} + m_{22} + \dots + m_{nn}) = 0.$$

Thus, either $m_{11} + m_{22} + \dots + m_{nn} = 0$ or $n = 2$. If $m_{11} + m_{22} + \dots + m_{nn} = 0$, then from (4.24), we get

$$m_{ii} = 0 \text{ for all } i \in \{1, \dots, n\}.$$

By the above equation and (4.23), we obtain $m_{ij} = 0$ for all $i, j \in \{1, \dots, n\}$, that implies that p is a totally geodesic point. If $n = 2$, then from (4.24), $2m_{11} = 2m_{22} = m_{11} + m_{22}$, that is, p is a totally umbilical point. Converse is trivial. \square

Lemma 4.2. *If $n > k \geq 2$ and $a_1, \dots, a_n \in \mathbb{R}$ are real numbers such that*

$$\left(\sum_{i=1}^n a_i \right)^2 = (n - k + 1) \left(\sum_{i=1}^n a_i^2 + a \right),$$

then

$$2 \sum_{1 \leq i < j \leq k} a_i a_j \geq a.$$

with equality holding iff

$$a_1 + a_2 + \dots + a_k = a_{k+1} = \dots = a_n.$$

Theorem 4.4. *Let M be an $(n + 1)$ -dimensional screen homothetic lightlike hypersurface with $\varphi > 0$ of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c , endowed with a semi-symmetric metric connection $\widetilde{\nabla}$ such that the vector field P is tangent to M . Then, for each point $p \in M$ and each non-degenerate k -plane section $\Pi_k \subset T_p M$ ($n > k \geq 2$), we have*

$$\begin{aligned} \tau_{S(TM)}(p) - \tau(\pi_k) &\geq (n - k) \left(\frac{\varphi n^2}{(n - k + 1)} \mu^2 + (n - k + 1)c - \lambda \right) \\ &\quad - \varphi \sum_{r=k}^n (m_{rr})^2 + 2(k - 1) \text{trace}(\alpha|_{\pi_k^\perp}). \end{aligned} \tag{4.25}$$

If the equality case of (4.25) satisfies for $p \in M$, thus M is minimal and the form of shape operator of M becomes

$$A_\xi^* = \begin{bmatrix} m_{11} & m_{12} & \cdot & \cdot & m_{1k} & & \\ m_{21} & m_{22} & \cdot & \cdot & m_{2k} & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \\ m_{k1} & m_{k2} & \cdot & \cdot & -\sum_{i=1}^{k-1} (m_{ii}) & & \\ & & 0 & & & & 0_{n-k} \end{bmatrix}. \tag{4.26}$$

Proof. One takes

$$\varepsilon = \tau_{S(TM)}(p) - n(n-1)c + 2(n-1)\lambda - \varphi \frac{n^2(n-k)}{(n-k+1)}\mu^2, \tag{4.27}$$

in (4.8), then we have

$$\varepsilon = \varphi \frac{n^2}{(n-k+1)}\mu^2 - \varphi \sum_{i,j=1}^n (m_{ij})^2.$$

Therefore, we can write

$$\left(\sum_{i=1}^n m_{ii}\right)^2 = (n-k+1) \left(\sum_{i=1}^n (m_{ii})^2 + \sum_{i \neq j=1}^n (m_{ij})^2 + \frac{\varepsilon}{\varphi}\right). \tag{4.28}$$

From Lemma 4.2 we get

$$2 \sum_{1 \leq i < j \leq k} m_{ii}m_{jj} \geq \sum_{i \neq j=1}^n (m_{ij})^2 + \frac{\varepsilon}{\varphi}. \tag{4.29}$$

Now, a non-degenerate plane section π_k spanned by $\{e_1, e_2, \dots, e_k\}$. Then one obtains

$$\begin{aligned} \tau(\pi_k) &= k(k-1)c - \sum_{i,j=1}^k (\alpha(e_i, e_i) + \alpha(e_j, e_j)) + \varphi \sum_{i,j=1}^k m_{ii}m_{jj} - (m_{ij})^2 \\ &= k(k-1)c - \sum_{i,j=1}^k (\alpha(e_i, e_i) + \alpha(e_j, e_j)) + \varphi \sum_{i=1}^k (m_{ii})^2 \\ &\quad + 2\varphi \sum_{1 \leq i < j \leq k} m_{ii}m_{jj} - \varphi \sum_{i,j=1}^k (m_{ij})^2 \\ &\geq k(k-1)c - 2(k-1) \sum_{i=1}^k \alpha(e_i, e_i) + \varepsilon + \sum_{i \neq j=1}^n (m_{ij})^2 - \varphi \sum_{i,j=1}^k (m_{ij})^2 \\ &\geq k(k-1)c - 2(k-1) \sum_{i=1}^k \alpha(e_i, e_i) + \varepsilon + \varphi \sum_{i,j=1}^n (m_{ij})^2 - \varphi \sum_{i=1}^n (m_{ii})^2 - \varphi \sum_{i,j=1}^k (m_{ij})^2 \\ &\geq k(k-1)c - 2(k-1) \sum_{i=1}^k \alpha(e_i, e_i) + \varepsilon + \varphi \sum_{i,j=k+1}^n (m_{ij})^2 - \varphi \sum_{i=k}^n (m_{ii})^2. \end{aligned} \tag{4.30}$$

We remark that

$$\sum_{i=1}^k \alpha(e_i, e_i) = \lambda - \text{trace}(\alpha|_{\pi_k^\perp}). \tag{4.31}$$

Using (4.27), (4.30) and (4.31) we get

$$\begin{aligned} \tau(\pi_k) &\geq k(k-1)c - 2(k-1) \left(\lambda - \text{trace}(\alpha|_{\pi_k^\perp})\right) - \varphi \sum_{i=k}^n (m_{ii})^2 \\ &\quad + \tau_{S(TM)}(p) - n(n-1)c + 2(n-1)\lambda - \varphi \frac{n^2(n-k)}{n-k+1}\mu^2. \end{aligned} \tag{4.32}$$

From (4.32) we have (4.25) and (4.26) which implies that M is minimal. □

Furthermore, the second fundamental form m and the screen second fundamental form D provide

$$\sum_{i,j=1}^n m_{ij}D_{ji} = \frac{1}{2} \left\{ \sum_{i,j=1}^n (m_{ij} + D_{ji})^2 - \sum_{i,j=1}^n (m_{ij})^2 + (D_{ji})^2 \right\} \tag{4.33}$$

and

$$\sum_{i,j=1}^n m_{ii}D_{jj} = \frac{1}{2} \left\{ \left(\sum_{i,j=1}^n m_{ii} + D_{jj} \right)^2 - \left(\sum_{i=1}^n m_{ii} \right)^2 - \left(\sum_{j=1}^n D_{jj} \right)^2 \right\}. \quad (4.34)$$

Theorem 4.5. Let M be an $(n + 1)$ -dimensional lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c , endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. Then, we have

(i)

$$\tau_{S(TM)}(p) \leq n(n - 1)c - 2(n - 1)\lambda + n\mu \text{trace}A_N + \frac{1}{2} \sum_{i,j=1}^n (m_{ij})^2 + (D_{ji})^2. \quad (4.35)$$

The equality case of (4.35) satisfies for all $p \in M$ iff either M is a screen homothetic lightlike hypersurface with $\varphi = -1$ or M is a totally geodesic lightlike hypersurface.

(ii)

$$\tau_{S(TM)}(p) \geq n(n - 1)c - 2(n - 1)\lambda + n\mu \text{trace}A_N - \frac{1}{2} \sum_{i,j=1}^n (m_{ij})^2 + (D_{ji})^2. \quad (4.36)$$

The equality case of (4.36) satisfies for all $p \in M$ iff either M is a screen homothetic lightlike hypersurface with $\varphi = 1$ or M is a totally geodesic lightlike hypersurface.

(iii) (4.35) and (4.36) with equalities iff p is a totally geodesic point.

Proof. From (4.7) and (4.33), we get

$$\tau_{S(TM)}(p) = n(n - 1)c - 2(n - 1)\lambda + \sum_{i,j=1}^n m_{ii}D_{jj} - \frac{1}{2} \sum_{i,j=1}^n (m_{ij} + D_{ji})^2 + \frac{1}{2} \sum_{i,j=1}^n (m_{ij})^2 + (D_{ji})^2 \quad (4.37)$$

which yields (4.35).

Since

$$\frac{1}{2}((m_{ij})^2 + (D_{ji})^2) = \frac{1}{4}(m_{ij} + D_{ji})^2 + \frac{1}{4}(m_{ij} - D_{ji})^2, \quad (4.38)$$

one obtains

$$\tau_{S(TM)}(p) = n(n - 1)c - 2(n - 1)\lambda + \sum_{i,j=1}^n m_{ii}D_{jj} + \frac{1}{2} \sum_{i,j=1}^n (m_{ij} - D_{ji})^2 - \frac{1}{2} \sum_{i,j=1}^n (m_{ij})^2 + (D_{ji})^2 \quad (4.39)$$

which implies (4.36). From (4.35), (4.36), (4.37) and (4.39) (i), (ii) and (iii) statements are easily obtained. \square

Thus we get the following corollary.

Corollary 4.1. Let M be an $(n + 1)$ -dimensional screen homothetic lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c , endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. Then, we have

(i)

$$\tau_{S(TM)}(p) \leq n(n - 1)c - 2(n - 1)\lambda + \varphi n^2 \mu^2 + \left(\frac{1 + \varphi^2}{2} \right) \sum_{i,j=1}^n (m_{ij})^2. \quad (4.40)$$

(ii)

$$\tau_{S(TM)}(p) \geq n(n - 1)c - 2(n - 1)\lambda + \varphi n^2 \mu^2 - \left(\frac{1 + \varphi^2}{2} \right) \sum_{i,j=1}^n (m_{ij})^2. \quad (4.41)$$

Theorem 4.6. Let M be an $(n + 1)$ -dimensional lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c , endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. Then, we derive

$$\begin{aligned} \tau_{S(TM)}(p) \leq & n(n - 1)c - 2(n - 1)\lambda + \frac{1}{2}(\text{trace}\bar{A})^2 - \frac{1}{2}(\text{trace}A_N)^2 \\ & - \frac{1}{4} \sum_{i,j=1}^n (m_{ij} + D_{ji})^2 + \frac{1}{4} \sum_{i,j=1}^n (m_{ij} - D_{ji})^2, \end{aligned} \quad (4.42)$$

where

$$\bar{A} = \begin{bmatrix} m_{11} + D_{11} & m_{12} + D_{21} & \cdot & \cdot & \cdot & m_{1n} + D_{n1} \\ m_{21} + D_{12} & m_{22} + D_{22} & \cdot & \cdot & \cdot & m_{2n} + D_{n2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ m_{n1} + D_{1n} & m_{n2} + D_{2n} & \cdot & \cdot & \cdot & m_{nn} + D_{nn} \end{bmatrix}. \tag{4.43}$$

For all $p \in M$ the equality case of (4.42) satisfies iff M is minimal.

Proof. From (4.7), (4.33) and (4.34) we obtain

$$\begin{aligned} \tau_{S(TM)}(p) &= n(n-1)c - 2(n-1)\lambda + \frac{1}{2} \sum_{i,j=1}^n (m_{ii} + D_{jj})^2 - \frac{1}{2} \left(\sum_{i=1}^n m_{ii} \right)^2 \\ &\quad - \frac{1}{2} \left(\sum_{j=1}^n D_{jj} \right)^2 - \frac{1}{2} \sum_{i,j=1}^n (m_{ij} + D_{ji})^2 + \frac{1}{2} \sum_{i,j=1}^n (m_{ij})^2 + (D_{ji})^2. \end{aligned} \tag{4.44}$$

From (4.38) we have

$$-\frac{1}{2} \sum_{i,j=1}^n (m_{ij} + D_{ji})^2 + \frac{1}{2} \sum_{i,j=1}^n (m_{ij})^2 + (D_{ji})^2 = -\frac{1}{4} \sum_{i,j=1}^n (m_{ij} + D_{ji})^2 + \frac{1}{4} \sum_{i,j=1}^n (m_{ij} - D_{ji})^2. \tag{4.45}$$

Using (4.45) in (4.44), we get

$$\begin{aligned} \tau_{S(TM)}(p) &= n(n-1)c - 2(n-1)\lambda + \frac{1}{2} \sum_{i,j=1}^n (m_{ii} + D_{jj})^2 - \frac{1}{2} \left(\sum_{i=1}^n m_{ii} \right)^2 \\ &\quad - \frac{1}{2} \left(\sum_{j=1}^n D_{jj} \right)^2 - \frac{1}{4} \sum_{i,j=1}^n (m_{ij} + D_{ji})^2 + \frac{1}{4} \sum_{i,j=1}^n (m_{ij} - D_{ji})^2. \end{aligned} \tag{4.46}$$

Assume the equality case of (4.42) is satisfied, then

$$\sum_i m_{ii} = 0.$$

Thus M is minimal. □

Thus we get the following corollary.

Corollary 4.2. Let M be an $(n + 1)$ -dimensional lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c , endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. Then, we get

$$\tau_{S(TM)}(p) \leq n(n-1)c - 2(n-1)\lambda + \left(\frac{2\varphi + 1}{2} \right) n^2 \mu^2 - \varphi \sum_{i=1}^n (m_{ij})^2. \tag{4.47}$$

For all $p \in M$ the equality case of (4.47) satisfies iff M is minimal.

Theorem 4.7. Let M be an $(n + 1)$ -dimensional screen homothetic lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c , endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. Then, we derive

$$\begin{aligned} \tau_{S(TM)}(p) &\leq n(n-1)c - 2(n-1)\lambda + \frac{(n-1)}{2n} (\text{trace } \bar{A})^2 - \frac{1}{2} (\text{trace } A_N)^2 \\ &\quad - \frac{1}{2} n^2 \mu^2 - \frac{1}{2} \sum_{i \neq j} (m_{ij} + D_{ji})^2 + \frac{1}{2} \sum_{i,j=1}^n (m_{ij})^2 + (D_{ji})^2, \end{aligned} \tag{4.48}$$

where \bar{A} is equal to (4.43).

For all $p \in M$ the equality case of (4.48) satisfies iff $n\mu = -\text{trace } A_N$.

Proof. From (4.44), we get

$$\begin{aligned} \tau_{S(TM)}(p) &= n(n-1)c - 2(n-1)\lambda + \frac{1}{2}(\text{trace}\bar{A})^2 - \frac{1}{2}(\text{trace}A_N)^2 - \frac{1}{2}n^2\mu^2 \\ &\quad - \frac{1}{2}\sum_{i=1}^n(m_{ii} + D_{ii})^2 - \frac{1}{2}\sum_{i \neq j}(m_{ij} + D_{ji})^2 + \frac{1}{2}\sum_{i,j=1}^n(m_{ij})^2 + (D_{ji})^2. \end{aligned} \tag{4.49}$$

Using Lemma 4.1 and equality case of (4.49), we have

$$\begin{aligned} \tau_{S(TM)}(p) &\leq n(n-1)c - 2(n-1)\lambda + \frac{1}{2}(\text{trace}\bar{A})^2 - \frac{1}{2}(\text{trace}A_N)^2 - \frac{1}{2}n^2\mu^2 \\ &\quad - \frac{1}{2n}\left(\sum_{i=1}^n m_{ii} + D_{ii}\right)^2 - \frac{1}{2}\sum_{i \neq j}(m_{ij} + D_{ji})^2 + \frac{1}{2}\sum_{i,j=1}^n(m_{ij})^2 + (D_{ji})^2 \end{aligned} \tag{4.50}$$

which implies (4.48). The equality case of (4.48) holds, then

$$m_{11} + D_{11} = m_{22} + D_{22} = \dots = m_{nn} + D_{nn}. \tag{4.51}$$

From (4.51) we obtain

$$\begin{aligned} (1-n)m_{11} + m_{22} + \dots + m_{nn} + (1-n)D_{11} + D_{22} + \dots + D_{nn} &= 0, \\ m_{11} + (1-n)m_{22} + \dots + m_{nn} + D_{11} + (1-n)D_{22} + \dots + D_{nn} &= 0, \\ &\vdots \\ &\vdots \\ &\vdots \\ m_{11} + m_{22} + \dots + (1-n)m_{nn} + D_{11} + D_{22} + \dots + (1-n)D_{nn} &= 0. \end{aligned}$$

Using last equations, we have

$$(n-1)^2(\text{trace}A_N + n\mu) = 0. \tag{4.52}$$

Because of $n \neq 1$, we get $n\mu = -\text{trace}A_N$. □

Thus we get the following corollary.

Corollary 4.3. *Let M be an $(n+1)$ -dimensional screen homothetic lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c , endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. Then, we have*

$$\tau_{S(TM)}(p) \leq n(n-1)c - 2(n-1)\lambda + \varphi n(n-1)\mu^2 - \frac{(1+\varphi^2)}{2}n\mu^2 - \varphi \sum_{i \neq j}^n(m_{ij})^2 + \frac{(1+\varphi^2)}{2} \sum_{i=1}^n(m_{ii})^2. \tag{4.53}$$

For all $p \in M$ the equality case of (4.53) satisfies iff either $\varphi = -1$ or M is minimal.

Theorem 4.8. *Let M be an $(n+1)$ -dimensional lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ of constant sectional curvature c , endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. Then*

$$\begin{aligned} \tau_{S(TM)}(p) &\geq n(n-1)c - 2(n-1)\lambda + \frac{1}{2}((\text{trace}\bar{A})^2 - (\text{trace}A_N)^2 - n(n-1)\mu^2) \\ &\quad + \frac{1}{2}\left(\sum_{i \neq j=1}^n(m_{ij})^2 - \sum_{i,j=1}^n(m_{ij} + D_{ji})^2 + \sum_{i,j=1}^n(D_{ji})^2\right). \end{aligned} \tag{4.54}$$

For all $p \in M$ the equality case of (4.54) satisfies iff p is a totally umbilical point.

Proof. Using (4.44), we get

$$\begin{aligned} \tau_{S(TM)}(p) &= n(n-1)c - 2(n-1)\lambda + \frac{1}{2}(\text{trace}\bar{A})^2 - \frac{1}{2}(\text{trace}A_N)^2 - \frac{1}{2}n^2\mu^2 \\ &\quad + \frac{1}{2}\sum_{i=1}^n(m_{ii})^2 + \frac{1}{2}\sum_{i \neq j=1}^n(m_{ij})^2 + \frac{1}{2}\sum_{i,j=1}^n(D_{ji})^2 - \frac{1}{2}\sum_{i,j=1}^n(m_{ij} + D_{ji})^2. \end{aligned} \tag{4.55}$$

Using Lemma 4.2 and equality case of (4.44), we have

$$\begin{aligned} \tau_{S(TM)}(p) \geq & n(n-1)c - 2(n-1)\lambda + \frac{1}{2}(\text{trace}\bar{A})^2 - \frac{1}{2}(\text{trace}A_N)^2 - \frac{1}{2}n^2\mu^2 \\ & - \frac{1}{2} \sum_{i,j=1}^n (m_{ij} + D_{ji})^2 + \frac{1}{2n} \left(\sum_{i=1}^n m_{ii}\right)^2 + \frac{1}{2} \sum_{i \neq j=1}^n (m_{ij})^2 + \frac{1}{2} \sum_{i,j=1}^n (D_{ji})^2 \end{aligned} \quad (4.56)$$

which implies (4.53). The equality case of (4.43) satisfies iff

$$m_{11} = \dots = m_{nn}$$

and the shape operator A_ξ^* becomes of the form

$$A_\xi^* = \begin{bmatrix} m_{11} & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & m_{11} & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & & \cdot & & & & \\ \cdot & & & \cdot & & & \\ \cdot & & & & \cdot & & \\ 0 & 0 & \cdot & \cdot & \cdot & m_{11} & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \end{bmatrix}, \quad (4.57)$$

which indicates that M is totally umbilical. Hence, the claim holds. □

Thus we get the following corollary.

Corollary 4.4. *Let M be an $(n + 1)$ -dimensional screen homothetic lightlike hypersurface of a Lorentzian space form $\tilde{M}(c)$ of constant sectional curvature c , endowed with a semi-symmetric metric connection $\tilde{\nabla}$. Then, we get*

$$\tau_{S(TM)}(p) \geq n(n-1)c - 2(n-1)\lambda + \frac{(2\varphi + 1)}{2}n^2\mu^2 - \frac{n(n-1)}{2}\mu^2 - \frac{(2\varphi + 1)}{2} \sum_{i,j=1}^n (m_{ij})^2. \quad (4.58)$$

For all $p \in M$ the equality case of (4.58) satisfies iff p is a totally umbilical point.

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